Qualitative study of functions (exercises with detailed solutions)

Exercise 1 Let \( f(x) = e^{2x} - 3e^x + 2 \).

a) Find the domain, the limits at the endpoints of the domain and the asymptotes. Find for which values of \( x \) \( f \) vanishes and study the sign of \( f \).

b) Find the monotonicity intervals, local and global minima and maxima of \( f \).

c) Find the convexity and concavity intervals and the inflection points of \( f \).

d) Draw a qualitative graph of \( f \).

e) Discuss the existence of solutions for the equation \( e^{2x} - 3e^x = \alpha \) where \( \alpha \in \mathbb{R} \).

\[ \text{dom}(f) = \mathbb{R} \text{ and } f(x) = (e^x - 1)(e^x - 2). \] Hence \( f \) vanishes when \( x = 0 \) and when \( x = \log 2 \). \( f > 0 \) if \( x > 0 \) or \( x > \log 2 \); \( f < 0 \) if \( 0 < x < \log 2 \). We have

\[ \lim_{x \to -\infty} f(x) = 2, \quad \lim_{x \to +\infty} f(x) = +\infty. \]

\( y = 2 \) is an horiz. asymp. as \( x \to -\infty \); As \( x \to +\infty \), \( f \) does not have any asymptote.

The first derivative of \( f \) is

\[ f'(x) = 2e^{2x} - 3e^x = e^x(2e^x - 3) \]

and it vanishes when \( x = \log(3/2) \), is negative when \( x < \log(3/2) \) (where \( f \) decreases) and positive when \( x > \log(3/2) \) (where \( f \) increases). Since \( f \) is continuous we deduce that \( x = \log \frac{3}{2} \) is a minimum.

The second derivative of \( f \) is

\[ f''(x) = 4e^{2x} - 3e^x = e^x(4e^x - 3), \]

hence \( x = \log(3/4) \) is an inflection point. \( f \) is concave if \( x < \log(3/4) \), convex if \( x > \log(3/4) \).

The minimum point we have found is an absolute minimum, while \( f \) does not have any maximum point (\( f \) is not bounded from above).

In order to answer to the last question, we remark that the minimal value of \( f \) is \(-1/4\). Let \( \beta = 2 + \alpha \), we have:

- the equation has no solutions if \( \beta < -\frac{1}{4} \) (that is if \( \alpha < -\frac{9}{4} \)),
- the equation has two solutions if \(-\frac{1}{4} < \beta < 2 \) \((-\frac{9}{4} < \alpha < 0)\),
- the equation has one solution if \( \beta \geq 2 \) or if \( \beta = -\frac{1}{4} \) \((\alpha \geq 0 \) or \( \alpha = -\frac{9}{4} \)).
Exercise 2 Let \( f(x) = x + \log(x^2 - 5x + 6) \).

a) Find the domain, the limits at the endpoints of the domain and the asymptotes.

b) Find the monotonicity intervals, local and global minima and maxima of \( f \).

c) Find the convexity and concavity intervals and the inflection points of \( f \).

d) Draw a qualitative graph of \( f \).

The function \( f \) is defined when \( x \in I \cup J \), where \( I = (-\infty, 2) \) and \( J = (3, +\infty) \). Furthermore, 
\[
\lim_{x \to -\infty} f(x) = -\infty, \quad \lim_{x \to 2^-} f(x) = -\infty, \quad \lim_{x \to 3^+} f(x) = -\infty, \quad \lim_{x \to +\infty} f(x) = +\infty
\]
indeed just in the first limit (in the others we simply replace) we have an indeterminate form and we can solve it as follows
\[
\lim_{x \to -\infty} x + \log(x^2 - 5x + 6) = \lim_{x \to -\infty} x + \log x^2 = \lim_{x \to -\infty} x + 2 \log x = \lim_{x \to -\infty} x(1 + 2(\log x)/x) = -\infty.
\]

\( f \) does not have any oblique asymptote.

We have 
\[
f'(x) = 1 + \frac{2x - 5}{x^2 - 5x + 6} = \frac{x^2 - 3x + 1}{x^2 - 5x + 6}, \quad \forall x \in I \cup J.
\]
x
\[
x^2 - 3x + 1 = 0 \text{ has 2 solutions, } x = (3 \pm \sqrt{5})/2, \text{ but only } x = (3 - \sqrt{5})/2 \in \text{dom}(f) \text{ Furthermore } f'(x) > 0 \text{ if } x < \frac{3 - \sqrt{5}}{2} \text{ or } x > 3; \quad f'(x) < 0 \text{ if } \frac{3 - \sqrt{5}}{2} < x < 2.
\]
Hence \( f \) has a (local) maximum at \( x = (3 - \sqrt{5})/2 \). The second derivative is
\[
f''(x) = \frac{-2x^2 + 10x - 13}{(x^2 - 5x + 6)^2}.
\]
\(-2x^2 + 10x - 13 < 0 \) for every \( x \in \mathbb{R} \). Then \( f \) does not have any inflection points and it is concave both on \( I \) and on \( J \) (but not on \( I \cup J \)).

In order to draw the graph of \( f \), we study the intersections of its graph with the \( x \)-axes. At the endpoints of \( J \) we know that \( f \) tends to \( -\infty \) and to \( +\infty \) respectively, and that \( f \) increases on the whole interval. Then the graph has exactly one intersection with the \( x \)-axes in \( J \).

On \( I \), since the limits at the endpoints are both \( -\infty \) we want to understand if \( f(x) > 0 \) for some \( x \in I \).

The level of the maximum point is difficult to compute, but \( f(0) = \log 6 > 0 \). Since \( f \) is continuous and has a unique maximum we deduce the existence of exactly two intersections between its graph and the \( x \)-axes in the interval \( I \).
Exercise 3 Let \( f(x) = \sqrt{1 + \log (2 - x^2)} \).

a) Find the domain of \( f \).

b) Find the monotonicity intervals, local and global minima and maxima of \( f \).

c) Draw a qualitative graph of \( f \).

d) prove that \( f \) is invertible on \( \text{dom}(f) \cap (-\infty, -1) \), find \( f^{-1} \) specifying its domain and range.

\[ \text{dom}(f) = [-\sqrt{2 - e^{-1}}, \sqrt{2 - e^{-1}}] \] is even and continuous on \( \text{dom}(f) \) (composition of continuous functions).

Since \( \text{dom}(f) \) is closed and bounded, from Weierstrass’ Theorem we deduce that \( f \) achieves its global maximum and minimum. \( f \) is differentiable in \( (-\sqrt{2 - e^{-1}}, \sqrt{2 - e^{-1}}) \) and

\[ f'(x) = \frac{x}{(x^2 - 2)\sqrt{1 + \log (2 - x^2)}}. \]

\( f'(x) = 0 \) if and only if \( x = 0 \). Furthermore since

\[ f'(x) > 0 \iff -\sqrt{2 - e^{-1}} < x < 0. \]

\( x = 0 \) is a maximum point for \( f \). \( f \) increases in \( (-\sqrt{2 - e^{-1}}, 0) \), decreases in \( (0, \sqrt{2 - e^{-1}}) \), hence at \( x = 0 \) \( f \) achieves its global maximum. The global minimum is achieved at two different points: \( x = \pm \sqrt{2 - e^{-1}} \). We remark that at the endpoints of its domain \( f \) is not differentiable, indeed

\[ \lim_{x \to -\sqrt{2 - e^{-1}}} f'(x) = -\infty, \quad \lim_{x \to -\sqrt{2 - e^{-1}}} f'(x) = +\infty. \]

In order to draw the graph of \( f \) we remark that \( f \) vanishes when \( x = \pm \sqrt{2 - e^{-1}} \) and that \( f(0) = \sqrt{1 + \log 2} \). If we call \( g \) the restriction of \( f \) to \( [-\sqrt{2 - e^{-1}}, -1] \) we have that \( g \) is injective, since strictly increasing. We can then invert \( g \); since

\[ \lim_{x \to -1} g(x) = 1 \quad \text{we have} \quad \mathcal{R}(g) = [0, 1). \]

Then

\[ \text{dom}(g^{-1}) = [0, 1), \quad \mathcal{R}(g^{-1}) = [-\sqrt{2 - e^{-1}}, -1]. \]

We now explicit \( x \) as a function of \( y \) in \( y = g(x) \), that is

\[ x = -\sqrt{2 - e^{-y^2-1}}. \]

The inverse function is then

\[ g^{-1}(x) = -\sqrt{2 - e^{-x^2}}. \]
Exercise 4 Let \( f(x) = \begin{cases} 
\frac{5 + 2\log |x|}{2 + \log |x|} & \text{if } x \neq 0 \\
0 & \text{if } x = 0.
\end{cases} \)

a) Find the domain, the limits at the endpoints of the domain and the asymptotes.
b) Find the monotonicity intervals, local and global minima and maxima of \( f \).
c) Find the convexity and concavity intervals and the inflection points of \( f \).
d) Draw a qualitative graph of \( f \).

\( \text{dom}(f) = (-\infty, -e^{-2}) \cup (-e^{-2}, e^{-2}) \cup (e^{-2}, +\infty) \) and \( f \) is even; then we just study it when \( x \geq 0 \). When \( x > 0 \) we have

\[
f(x) = \frac{5 + 2\log x}{2 + \log x}
\]

\( f(x) = 0 \) when \( x = e^{-\frac{3}{2}} \). \( f(x) > 0 \) when \( 0 \leq x < e^{-\frac{3}{2}} \) or \( x > e^{-2} \), \( f(x) < 0 \) when \( e^{-\frac{3}{2}} < x < e^{-2} \). The limits of \( f \) are

\[
\lim_{x \to -\infty} f(x) = 2 \quad \Rightarrow \quad y = 2 \text{ is an horiz. asymp.,}
\]

\[
\lim_{x \to (e^{-2})^\pm} f(x) = \pm\infty \quad \Rightarrow \quad x = e^{-2} \text{ is a vert. asymp.}
\]

Furthermore

\[
\lim_{x \to 0^+} f(x) = 2
\]

and \( f \) is continuous from the right at 0: since \( f \) is even we get that it is continuous at 0. Then \( f \) is continuous on its dom

\( f \) is differentiable whenever \( x > 0 \) (\( x \neq e^{-2} \)), and

\[
f'(x) = -\frac{1}{x(2 + \log x)^2}.
\]

\( f \) is not differentiable at \( x = 0 \) (cusp), indeed

\[
\lim_{x \to 0^+} f'(x) = -\infty, \quad \lim_{x \to 0^-} f'(x) = +\infty.
\]

\( f'(x) \neq 0 \) hence the extremal points belong to the set of where \( f \) is not differentiable. Hence the unique (possible) extremal point is \( x = 0 \). \( f'(x) > 0 \) for every \( x > 0 \) (\( x \neq e^{-2} \)) and \( f \) increases in \((0, e^{-2})\) and in \((e^{-2}, +\infty)\). \( x = 0 \) is a local maximum. \( f' \) is differentiable whenever \( x > 0 \) (\( x \neq e^{-2} \)) and

\[
f''(x) = \frac{\log x + 4}{x^2(2 + \log x)^3}.
\]

\( f''(x) = 0 \) when \( x = e^{-4} \) and we have

\[
f''(x) > 0 \iff 0 < x < e^{-4}, \quad x > e^{-2}.
\]

\( f \) is convex in \((0, e^{-4})\) and in \((-e^{-2}, +\infty)\), \( f \) is concave in \((e^{-4}, e^{-2})\). \( x = e^{-4} \) is an inflection point.
Exercise 5 Let \( f(x) = \frac{1}{2}|x + 1| - \arctan |x| \).

a) Find the domain, the limits at the endpoints of the domain and the asymptotes.

b) Find the monotonicity intervals, local and global minima and maxima of \( f \). At which points is \( f \) not differentiable?

c) Find the convexity and concavity intervals and the inflection points of \( f \).

d) Draw a qualitative graph of \( f \). Discuss the sign of the function and the existence of intersection points between the graph of \( f \) and the \( x \)-axis.

\( \text{dom}(f) = \mathbb{R} \) and \( \lim_{x \to \pm \infty} f(x) = +\infty \).

\( y = (-x - 1 - \pi)/2 \) is obl. as. at \( -\infty \) and \( y = (x + 1 - \pi)/2 \) at \( +\infty \). In order to compute the derivative and to study its sign we consider separately three situations:

1. \( f(x) = -\frac{x + 1}{2} + \arctan x \) and \( f'(x) = \frac{1 - x^2}{2(1 + x^2)} \) when \( x \in I = (-\infty, -1) \); \( f'(x) < 0 \) for every \( x \in I \);

2. \( f(x) = \frac{x + 1}{2} + \arctan x \) and \( f'(x) = \frac{x^2 + 3}{2(1 + x^2)} \) when \( x \in J = (-1, 0) \); \( f'(x) > 0 \) for every \( x \in J \);

3. \( f(x) = -\frac{x + 1}{2} - \arctan x \) and \( f'(x) = \frac{x^2 - 1}{2(1 + x^2)} \) when \( x \in K = (0, +\infty) \); \( f'(x) < 0 \) when \( x \in (0, 1) \) and \( f'(x) > 0 \) when \( x > 1 \).

In particular, \( f'(x) = 0 \) when \( x = 1 \), where \( f \) has a minimum. We remark that

\[
\lim_{x \to -1^-} f'(x) = 0, \quad \lim_{x \to -1^+} f'(x) = 1
\]

while

\[
\lim_{x \to 0^-} f'(x) = \frac{3}{2}, \quad \lim_{x \to 0^+} f'(x) = \frac{-3}{2}
\]

At \( x = -1 \) and at \( x = 0 \) the function is not differentiable. Since \( f \) is continuous, from its monotonicity we deduce that \( x = -1 \) is a minimum and \( x = 0 \) is a maximum. The second derivative is

\[
f''(x) = \frac{-2x}{(1 + x^2)^2} \quad \text{when} \ x \in I \cup J, \quad f''(x) = \frac{2x}{2(1 + x^2)^2} \quad \text{when} \ x \in K.
\]

Hence \( f \) is convex separately the 3 intervals \( I, J, K \) and it does not have any inflection points. Since \( f(1) = 1 - \frac{\pi}{4} > 0 \) and \( f(-1) = -\frac{\pi}{4} < 0 \) there exist \( x' \in I \) and \( x'' \in J \) where \( f \) vanishes. \( f \) is positive if \( x < x' \) or if \( x > x'' \); \( f \) is negative if \( x' < x < x'' \).

From the graph we observe that the minimum at \( x = -1 \) is global. \( f \) does not admit any global maximum.