TIME-FREQUENCY ANALYSIS OF FOURIER INTEGRAL OPERATORS

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Abstract. We use time-frequency methods for the study of Fourier Integral operators (FIOs). In this paper we shall show that Gabor frames provide very efficient representations for a large class of FIOs. Indeed, similarly to the case of shearlets and curvelets frames [6, 27], the matrix representation of a Fourier Integral Operator with respect to a Gabor frame is well-organized. This is used as a powerful tool to study the boundedness of FIOs on modulation spaces. As special cases, we recapture boundedness results on modulation spaces for pseudo-differential operators with symbols in $M^{-1}_{\infty,1}$ [25], for some unimodular Fourier multipliers [2] and metaplectic operators [10, 23].

1. Introduction

Fourier Integral Operators (FIOs) are a mathematical tool to study variety of problems arising in partial differential equations. Originally introduced by Lax [33] for the construction of parametrices in the Cauchy problem for hyperbolic equations, they have been widely employed to represent solutions to Cauchy problems, in the framework of both pure and applied mathematics (see, e.g., the papers [5, 6, 12, 13, 27, 30], the books [31, 35, 36] and references therein). In particular, they were employed by Helffer and Robert [28, 29] to study the spectral property of a class of globally elliptic operators, generalizing the harmonic oscillator of the Quantum Mechanics. The Fourier Integral operators we work with, possess a phase function similar to those of [28, 29]. A simple example is the resolvent of the Cauchy problem for the Schrödinger equation with a quadratic Hamiltonian.

For a given function $f$ on $\mathbb{R}^d$ the Fourier Integral Operator (FIO) $T$ with symbol $\sigma$ and phase $\Phi$ on $\mathbb{R}^{2d}$ can be formally defined by

$$Tf(x) = \int_{\mathbb{R}^d} e^{2\pi i \Phi(x,\eta)} \sigma(x,\eta) \hat{f}(\eta) d\eta.$$ 

The phase function $\Phi(x,\eta)$ is smooth on $\mathbb{R}^{2d}$, fulfills the estimates

$$|\partial^\alpha_z \Phi(z)| \leq C_\alpha, \quad |\alpha| \geq 2, \quad z \in \mathbb{R}^{2d},$$

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and the nondegeneracy condition
\[
|\det \partial_{x,\eta}^2 \Phi(x,\eta)| \geq \delta > 0, \quad (x,\eta) \in \mathbb{R}^{2d}.
\]

The symbol \(\sigma\) on \(\mathbb{R}^{2d}\) satisfies
\[
|\partial^\alpha z \sigma(z)| \leq C_\alpha, \quad |\alpha| \leq N, \quad \text{a.e. } z \in \mathbb{R}^{2d},
\]
for a fixed \(N > 0\) (in the sequel we shall work also with rougher symbols).

The first goal of this paper is to rephrase the operator \(T\) in terms of time-frequency analysis (see Gröchenig [23] and the next Section 2 for a review of the time-frequency methods.) Denoting \(T_x f(t) = f(t-x),\ M_\eta f(t) = e^{2\pi i t \eta} f(t)\), for \(\alpha, \beta > 0,\ g \in L^2(\mathbb{R}^d)\), the set of time-frequency shifts \(\mathcal{G}(g, \alpha, \beta) = \{g_{m,n} := M_\eta T_x g\} \text{ with } (m,n) \in \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d\), is a Gabor frame if there exist positive constants \(A,B > 0\) such that
\[
(3) \quad A \|f\|_{L^2} \leq \sum_{m,n} |\langle f, T_{m} M_{n} g \rangle| \leq B \|f\|_{L^2}, \quad \forall f \in L^2(\mathbb{R}^d).
\]

In Section 3 we show that the matrix representation of a FIO \(T\) with respect to a Gabor frame with \(g \in \mathcal{S}(\mathbb{R}^d)\) is well-organized (similarly to frames of curvelets and shearlets [6, 27]), provided that the symbol \(\sigma\) satisfies the decay estimate for every \(N > 0\) (see Theorem 3.3):

**Theorem 1.** For each \(N > 0\), there exists a constant \(C_N > 0\) such that
\[
|\langle Tg_{m,n}, g_{m',n'} \rangle| \leq C_N \langle \chi(m,n) - (m',n') \rangle^{-2N},
\]
where \(\chi\) is the canonical transformation generated by \(\Phi\).

In the special case of pseudodifferential operators such an almost diagonalization was already obtained in [24, 34]. Indeed, notice that pseudodifferential operators correspond to the phase \(\Phi(x,\eta) = x\eta\) and canonical transformation \(\chi(y,\eta) = (y,\eta)\).

As a relevant byproduct of the results of Section 3, we study the boundedness properties of the operator \(T\) on the so-called modulation spaces (Section 4 and 5). To define them, we fix a non-zero Schwartz function \(g\) and consider the short-time Fourier Transform \(V_g f\) of a function \(f\) on \(\mathbb{R}^d\) with respect to \(g\)
\[
V_g f(x,\eta) = \langle f, M_\eta T_x g \rangle = \int_{\mathbb{R}^d} f(t) g(t-x) e^{-2\pi i t \eta} dt
\]
which provides a time-frequency representation of \(f\). The (unweighted) modulation space \(M^{p,q}\) is the closure of the Schwartz class with respect to the norm
\[
\|f\|_{M^{p,q}} = \|V_g f\|_{L^{p,q}} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_g f(x,\eta)|^p \, dx \right)^{q/p} \, d\eta \right)^{1/p}
\]
(with appropriate modifications when \( p = \infty \) or \( q = \infty \)). In particular, when \( p = q \) we simply write \( M^{p,p} = M^p \), see Subsection 2.2 for exhaustive definitions and properties.

These spaces were introduced by Feichtinger [17] and have become canonical for both time-frequency and phase-space analysis [18], most recent employment being the study of PDEs [2, 3, 38, 39, 40].

If \( g \in M^1 \), and the Gabor frame \( \{ T_m M_n g; (m, n) \in \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d \} \) is a tight frame, namely (3) holds with \( A = B \), then it extends to a Banach frame for the modulation spaces \( M^{p,q}(\mathbb{R}^{2d}) \), with the norm equivalence
\[
\| f \|_{M^{p,q}} \asymp \| \langle f, T_m M_n g \rangle_{m,n} \|_{l^{p,q}}.
\]

Then, boundedness of the FIO \( T \) on \( M^{p,q} \) is equivalent to that of the infinite matrix \( \langle T g_m n, g_{m',n'} \rangle \) on the spaces of sequences \( l^{p,q} \).

Whence, the estimates (4) readily give (see Theorem 4.1 for a more general version):

**Theorem 2.** For \( N > d, 1 \leq p < \infty \), the Fourier integral operator \( T \), with symbol \( \sigma \) and phase \( \Phi \) as above, extends to a continuous operator on \( M^p \).

(In the case \( p = \infty \), the space \( M^\infty \) is replaced by the closure of the Schwartz function with respect to \( \| \cdot \|_{M^\infty} \).)

The continuity property of a FIO \( T \) on \( M^{p,q} \), with \( p \neq q \), fails in general. Indeed, an example is provided by the operator \( T f(x) = e^{\pi i |x|^2} f(x) \), corresponding to \( \Phi(x, \eta) = x \eta + |x|^2/2 \), \( \sigma \equiv 1 \), which is bounded on \( M^{p,q} \) if and only if \( p = q \) (see Proposition 7.1).

Hence, we introduce a new condition on the phase \( \Phi \), namely that the map \( x \mapsto -\nabla_x \Phi(x, \eta) \) has a range of finite diameter, uniformly with respect to \( \eta \), that allows us to get the boundedness on \( M^{p,q} \) (Theorem 5.2):

**Theorem 3.** For \( N > d, 1 \leq p, q < \infty \), the Fourier integral operator \( T \), with symbol \( \sigma \) and phase \( \Phi \) as above, and such that
\[
\sup_{x, x', \eta} |\nabla_x \Phi(x, \eta) - \nabla_x \Phi(x', \eta)| < \infty,
\]
extends to a continuous operator on \( M^{p,q} \).

(In the case \( p = \infty \) or \( q = \infty \), the space \( M^{p,q} \) is replaced by the closure of the Schwartz function with respect to \( \| \cdot \|_{M^{p,q}} \).)

As a particular case, we recapture recent boundedness results of unimodular Fourier multipliers [2] (see Example 5.3).

With respect to Theorem 2, here the proof is more delicate and combines the estimate (4) with a generalized version of Schur’s Test (Proposition 5.1).
To have a simple idea of the possible applications of Theorems 1, 2 and 3, consider the Cauchy problem

\[
\begin{cases}
i \frac{\partial u}{\partial t} + Hu = 0 \\
u(0, x) = u_0(x)
\end{cases}
\]

where $H$ is the Weyl quantization of a quadratic form on $\mathbb{R}^d \times \mathbb{R}^d$ (see, e.g., [10, 21]). Simple examples are $H = -\frac{1}{4\pi} \Delta + \pi |x|^2$, or $H = -\frac{1}{4\pi} \Delta - \pi |x|^2$ (see [4]). The solution to (5) is a one-parameter family of FIOs:

\[u(t, x) = e^{itH}u_0,\]

with symbol $\sigma \equiv 1$ and a phase given by a quadratic form $\Phi(x, \eta)$, satisfying trivially the preceding assumptions (1) and (2) (see [21] for details). We address to Section 7 for a debited study of such operators.

Finally, Section 6 presents a variant of Theorem 1, cf. (39), and a generalization of Theorem 2, cf. Theorem 6.1, to the case of FIOs $T$ with symbols in the modulation space $M^{\infty, 1}$. This generalizes the known boundedness results on $M^p$ of pseudodifferential operators with symbols in $M^{\infty, 1}$ [25], and intersects a previous result of Boulkhemair [5] on $L^2$ boundedness of FIOs. We address also to recent contribution [8], where the continuity and Schatten-von Neumann properties of similar operators when acting on $L^2$ are proved.

**Notation.** We define $|t|^2 = t \cdot t$, for $t \in \mathbb{R}^d$, and $xy = x \cdot y$ is the scalar product on $\mathbb{R}^d$.

The Schwartz class is denoted by $\mathcal{S}(\mathbb{R}^d)$, the space of tempered distributions by $\mathcal{S}'(\mathbb{R}^d)$. We use the brackets $\langle f, g \rangle$ to denote the extension to $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d)$ of the inner product $\langle f, g \rangle = \int f(t)g(t)dt$ on $L^2(\mathbb{R}^d)$. The Fourier transform is normalized to be $\hat{f}(\eta) = \mathcal{F}f(\eta) = \int f(t)e^{-2\pi i \eta t}dt$, the involution $g^\ast$ is $g^\ast(t) = \overline{g(-t)}$ and the inverse Fourier transform is $\hat{f}(\eta) = \mathcal{F}^{-1}f(\eta) = \hat{f}(-\eta)$.

Translation and modulation (time and frequency shifts) are defined, respectively, by

\[T_x f(t) = f(t - x) \quad \text{and} \quad M_\eta f(t) = e^{2\pi i \eta t}f(t).\]

We have the formulas $(T_x f)' = M_{-x} \hat{f}$, $(M_\eta f)' = T_\eta \hat{f}$, and $M_\eta T_x = e^{2\pi i \eta x}T_x M_\eta$. For $\alpha = (\alpha_1, \ldots, \alpha_d), \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}_+^d$, recall the multi-index notation $D^\alpha$ and $X^\beta$ for the operators of differentiation and multiplication

\[D^\alpha f = \prod_{j=1}^d \partial_{t_j}^{\alpha_j} f \quad \text{and} \quad X^\beta f(t) = \prod_{j=1}^d t_j^{\beta_j} f(t),\]
where \( t = (t_1, \ldots, t_d) \). We write \( dx \wedge d\xi = \sum_{j=1}^d dx_j \wedge d\xi_j \) for the canonical symplectic 2-form.

The spaces \( l^{p,q}_\mu = l^q |\mu|_p \), with weight \( \mu \), are the Banach spaces of sequences \( \{a_{m,n}\}_{m,n} \) on some lattice, such that

\[
\|a_{m,n}\|_{l^{p,q}_\mu} := \left( \sum_n \left( \sum_m |a_{m,n}|^{p}\mu(m,n)^p \right)^{q/p} \right)^{1/q} < \infty
\]

(with obvious changes when \( p = \infty \) or \( q = \infty \)).

We denote by \( c_0 \) the space of sequences vanishing at infinity. Throughout the paper, we shall use the notation \( A \lesssim B \) to indicate \( A \leq cB \) for a suitable constant \( c > 0 \), whereas \( A \asymp B \) if \( c^{-1}B \leq A \leq cB \) for a suitable \( c > 0 \).

2. Time-Frequency Methods

2.1. Short-Time Fourier Transform (STFT). The short-time Fourier transform (STFT) of a distribution \( f \in S'(\mathbb{R}^d) \) with respect to a non-zero window \( g \in S(\mathbb{R}^d) \) is

\[
V_g f(x, \eta) = \langle f, M_\eta T_x g \rangle = \int_{\mathbb{R}^d} f(t) \overline{g(t-x)} e^{-2\pi i \eta t} \, dt.
\]

The STFT \( V_g f \) is defined on many pairs of Banach spaces. For instance, it maps \( L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \) into \( L^2(\mathbb{R}^{2d}) \) and \( S(\mathbb{R}^d) \times S(\mathbb{R}^d) \) into \( S(\mathbb{R}^{2d}) \). Furthermore, it can be extended to a map from \( S'(\mathbb{R}^d) \times S'(\mathbb{R}^d) \) into \( S'(\mathbb{R}^{2d}) \).

We now recall the following inequality [23, Lemma 11.3.3], which is useful when one needs to change windows.

**Lemma 2.1.** Let \( g_0, g_1, \gamma \in S(\mathbb{R}^d) \) such that \( \langle \gamma, g_1 \rangle \neq 0 \) and let \( f \in S'(\mathbb{R}^d) \). Then,

\[
|V_{g_0} f(x, \eta)| \leq \frac{1}{|\langle \gamma, g_1 \rangle|} (|V_{g_1} f| * |V_{g_0} \gamma|)(x, \eta),
\]

for all \( (x, \eta) \in \mathbb{R}^{2d} \).

2.2. Modulation Spaces. The modulation space norms are a measure of the joint time-frequency distribution of \( f \in S' \). For their basic properties we refer, for instance, to [23, Ch. 11-13] and the original literature quoted there.

For the quantitative description of decay properties, we use weight functions on the time-frequency plane. In the sequel \( v \) will always be a continuous, positive, even, submultiplicative weight function (in short, a submultiplicative weight), hence \( v(0) = 1, v(z) = v(-z) \), and \( v(z_1 + z_2) \leq v(z_1)v(z_2) \), for all \( z, z_1, z_2 \in \mathbb{R}^{2d} \). A positive, weight function \( \mu \) on \( \mathbb{R}^{2d} \) belongs to \( M_v \), that is, is \( v \)-moderate if \( \mu(z_1 + z_2) \leq C_v(z_1)\mu(z_2) \) for all \( z_1, z_2 \in \mathbb{R}^{2d} \).
For our investigation of FIOs we will mostly use the polynomial weights defined by
\[ v_s(z) = v_s(x, \eta) = (1 + |x|^2 + |\eta|^2)^{s/2}, \quad z = (x, \eta) \in \mathbb{R}^{2d}. \]

Given a non-zero window \( g \in \mathcal{S}(\mathbb{R}^d) \), \( \mu \in \mathcal{M}_v \), and \( 1 \leq p, q \leq \infty \), the modulation space \( M_{\mu}^{p,q}(\mathbb{R}^d) \) consists of all tempered distributions \( f \in \mathcal{S}'(\mathbb{R}^d) \) such that \( V_g f \in L_{\mu}^{p,q}(\mathbb{R}^{2d}) \) (weighted mixed-norm spaces). The norm on \( M_{\mu}^{p,q} \) is
\[
\|f\|_{M_{\mu}^{p,q}} = \|V_g f\|_{L_{\mu}^{p,q}} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_g f(x, \eta)|^p \mu(x, \eta)^p \, dx \right)^{q/p} \, d\eta \right)^{1/p}
\]
(with obvious changes when \( p = \infty \) or \( q = \infty \)). If \( p = q \), we write \( M_{\mu}^p \) instead of \( M_{\mu}^{p,p} \), and if \( \mu(z) \equiv 1 \) on \( \mathbb{R}^{2d} \), then we write \( M_{\mu}^{p,q} \) and \( M_{\mu}^p \) for \( M_{\mu}^{p,q} \) and \( M_{\mu}^{p,p} \) respectively.

Then \( M_{\mu}^{p,q}(\mathbb{R}^d) \) is a Banach space whose definition is independent of the choice of the window \( g \). Moreover, if \( \mu \in \mathcal{M}_v \) and \( g \in M_1^1 \setminus \{0\} \), then \( \|V_g f\|_{L_{\mu}^{p,q}} \) is an equivalent norm for \( M_{\mu}^{p,q}(\mathbb{R}^d) \) (see [23, Thm. 11.3.7]):
\[
\|f\|_{M_{\mu}^{p,q}} \asymp \|V_g f\|_{L_{\mu}^{p,q}}.
\]

2.3. Wiener amalgam spaces. For a detailed treatment we refer to [15, 14, 16, 20, 22].

Let \( g \in D(\mathbb{R}^{2d}) \) be a test function that satisfies \( \sum_{(k,l) \in \mathbb{Z}^{2d}} T_{(k,l)} g \equiv 1 \). Let \( X(\mathbb{R}^{2d}) \) be a Banach space of functions invariant under translations and with the property that \( D \cdot X \subset X \), e.g., \( L^p, \mathcal{F}L^p \), or \( L^{p,q} \). Then the Wiener amalgam space \( W(X, L_{\mu}^{p,q}) \) with local component \( X \) and global component \( L_{\mu}^{p,q} \) is defined as the space of all functions or distributions for which the norm
\[
\|f\|_{W(X, L_{\mu}^{p,q})} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \|f \cdot T_{(z_1,z_2)} g\|_X^p \mu(z_1, z_2)^p \, dz_1 \right)^{q/p} \, dz_2 \right)^{1/q}
\]
is finite. Equivalently, \( f \in W(X, L_{\mu}^{p,q}) \) if and only if
\[
\left( \sum_{l \in \mathbb{Z}^d} \left( \sum_{k \in \mathbb{Z}^d} \|f \cdot T_{(k,l)} g\|_X^p \mu(k, l)^p \right)^{q/p} \right)^{1/q} < \infty.
\]

It can be shown that different choices of \( g \in D \) generate the same space and yield equivalent norms. In the sequel we shall use the inclusions relations between Wiener amalgam spaces: if \( B_1 \hookrightarrow B_2 \) and \( C_1 \hookrightarrow C_2 \),
\[
W(B_1, C_1) \hookrightarrow W(B_2, C_2).
\]

We now recall the following regularity property of the STFT [9, Lemma 4.1]:
**Lemma 2.2.** Let $1 \leq p, q \leq \infty$, $\mu \in \mathcal{M}_v$. If $f \in M_{\mu}^{p,q}(\mathbb{R}^d)$ and $g \in M_{\nu}^1(\mathbb{R}^d)$, then $V_g f \in W(\mathcal{F}L^1, L_{\mu}^{p,q})(\mathbb{R}^{2d})$ with norm estimate

$$\|V_g f\|_{W(\mathcal{F}L^1, L_{\mu}^{p,q})} \lesssim \|f\|_{M_{\mu}^{p,q}} \|\mu\|_{M_{\nu}^1}.$$  

We also give a slight generalization of [23, Proposition 11.1.4] and its subsequent Remark.

**Proposition 2.1.** Let $\mathcal{X}$ be a separated sampling set in $\mathbb{R}^{2d}$, that is, there exists $\delta > 0$, such that $\inf_{x,y \in \mathcal{X}, x \neq y} |x - y| \geq \delta$. Then there exists a constant $C > 0$ such that, if $F \in W(L^{\infty}, L_{\mu}^{p,q})$ is any function everywhere defined on $\mathbb{R}^{2d}$ and lower semi-continuous, then the restriction $F|_{\mathcal{X}}$ is in $L_{\mu}^{p,q}$, where $\mu = \mu|_{\mathcal{X}}$, and

$$\|F|_{\mathcal{X}}\|_{L_{\mu}^{p,q}} \leq C\|F\|_{W(L^{\infty}, L_{\mu}^{p,q})}.$$  

**Proof.** One uses the arguments of [23, Proposition 11.1.4] and its subsequent Remark. We just shall highlight the key points that make those arguments to work under our assumptions.

First of all, if $(r, s) \in \mathbb{Z}^{2d}$ and $\mathcal{X}$ is separated, then the number of sampling points of $\mathcal{X}$ in $(r, s) + [0, 1]^{2d}$ is bounded independently of $(r, s)$.

Secondly, for $x \in \mathcal{X}$ such that $x \in (r, s) + [0, 1]^{2d}$,

$$|F(x)|\mu(x) \leq C\|F \cdot T_{(r,s)}\chi_{[0,1]^{2d}}\|_{L^{\infty}}\mu(r, s),$$

Indeed, since $F$ is everywhere defined on $\mathbb{R}^{2d}$ and lower semi-continuous we have $\sup |f| = \text{ess sup}|f|$ on every box, whereas $\mu(x) \leq C\mu(r, s)$ because $\mu$ is $v$-moderate and $v$ is bounded on $[0, 1]^{2d}$ (cf. [23, Lemma 11.1.1]).

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**2.4. Gabor frames.** Fix a function $g \in L^2(\mathbb{R}^d)$ and a lattice $\Lambda = \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d$, for $\alpha, \beta > 0$. For $(m, n) \in \Lambda$, define $g_{m,n} := M_{m}T_{n}g$. The set of time-frequency shifts $\mathcal{G}(g, \alpha, \beta) = \{g_{m,n}, (m, n) \in \Lambda\}$ is called Gabor system. Associated to $\mathcal{G}(g, \alpha, \beta)$ we define the coefficient operator $C_g$, which maps functions to sequences as follows:

$$\left(C_{g} f\right)_{m,n} = \left(C_{g}^{\alpha, \beta} f\right)_{m,n} := \langle f, g_{m,n} \rangle, \quad (m, n) \in \Lambda,$$

the synthesis operator

$$D_g c = D_g^{\alpha, \beta} c = \sum_{(m,n) \in \Lambda} c_{m,n} T_{m} M_{n} g, \quad c = \{c_{m,n}\}_{(m,n) \in \Lambda}$$

and the Gabor frame operator

$$S_g f = S_g^{\alpha, \beta} f := D_g S_g f = \sum_{(m,n) \in \Lambda} \langle f, g_{m,n} \rangle g_{m,n}.$$  

The set $\mathcal{G}(g, \alpha, \beta)$ is called a Gabor frame for the Hilbert space $L^2(\mathbb{R}^d)$ if $S_g$ is a bounded and invertible operator on $L^2(\mathbb{R}^d)$. Equivalently, $C_g$ is bounded from $L^2(\mathbb{R}^d)$ to $L^2(\alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d)$ with closed range, i.e., $\|f\|_{L^2} \asymp \|C_g f\|_{L^2}$. If $\mathcal{G}(g, \alpha, \beta)$ is a
Gabor frame for \( L^2(\mathbb{R}^d) \), then the so-called \textit{dual window} \( \gamma = S_g^{-1} g \) is well-defined and the set \( \mathcal{G}(\gamma, \alpha, \beta) \) is a frame (the so-called canonical dual frame of \( \mathcal{G}(g, \alpha, \beta) \)). Every \( f \in L^2(\mathbb{R}^d) \) possesses the frame expansion

\[
 f = \sum_{(m,n) \in \Lambda} \langle f, g_{m,n} \rangle g_{m,n} = \sum_{(m,n) \in \Lambda} \langle f, \gamma_{m,n} \rangle \gamma_{m,n}
\]

with unconditional convergence in \( L^2(\mathbb{R}^d) \), and norm equivalence:

\[
 \| f \|_{L^2} \asymp \| C_g f \|_{l^2} \asymp \| C_\gamma f \|_{l^2}.
\]

This result is contained in [23, Proposition 5.2.1]. In particular, if \( \gamma = g \) and \( \| g \|_{L^2} = 1 \) the frame is called \textit{normalized tight} Gabor frame and the expansion (9) reduces to

\[
 f = \sum_{(m,n) \in \Lambda} \langle f, g_{m,n} \rangle g_{m,n}.
\]

If we ask for more regularity on the window \( g \), then the previous result can be extended to suitable Banach spaces, as shown below [19, 26].

**Theorem 2.2.** Let \( \mu \in \mathcal{M}_v \), \( \mathcal{G}(g, \alpha, \beta) \) be a normalized tight Gabor frame for \( L^2(\mathbb{R}^d) \), with lattice \( \Lambda = \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d \), and \( g \in M^1_v \). Define \( \tilde{\mu} = \mu_{|\Lambda} \).

(i) For every \( 1 \leq p, q \leq \infty \), \( C_g : M^{p,q}_\mu \to l^{p,q}_{\tilde{\mu}} \) and \( D_g : l^{p,q}_{\tilde{\mu}} \to M^{p,q}_\mu \) continuously and, if \( f \in M^{p,q}_\mu \), then the Gabor expansions (10) converge unconditionally in \( M^{p,q}_\mu \) for \( 1 \leq p, q < \infty \) and all weight \( \mu \), and weak*-\( M^\infty_\mu \) unconditionally if \( p = \infty \) or \( q = \infty \).

(ii) The following norms are equivalent on \( M^{p,q}_\mu \):

\[
\| f \|_{M^{p,q}_\mu} \asymp \| C_g f \|_{l^{p,q}_{\tilde{\mu}}}.
\]

We also establish the following properties. Denote by \( \hat{M}^{p,q}_\mu \) the closure of the Schwartz class in \( M^{p,q}_\mu \). Hence, \( \hat{M}^{p,q}_\mu = M^{p,q}_\mu \) if \( p < \infty \) and \( q < \infty \). Also, denote by \( \hat{l}^{p,q}_{\tilde{\mu}} \) the closure of the space of eventually zero sequences in \( l^{p,q}_{\tilde{\mu}} \). Hence \( \hat{l}^{p,q}_{\tilde{\mu}} = l^{p,q}_{\tilde{\mu}} \) if \( p < \infty \) and \( q < \infty \).

**Theorem 2.3.** Under the assumptions of Theorem 2.2, for every \( 1 \leq p, q \leq \infty \) the operator \( C_g \) is continuous from \( \hat{M}^{p,q}_\mu \) into \( l^{p,q}_{\tilde{\mu}} \), whereas the operator \( D_g \) is continuous from \( \hat{l}^{p,q}_{\tilde{\mu}} \) into \( \hat{M}^{p,q}_\mu \).

**Proof.** Since \( C_g \) is continuous from \( M^{p,q}_\mu \) into \( l^{p,q}_{\tilde{\mu}} \) it suffices to verify that, if \( f \) is a Schwartz function then \( C_g(f) \in \hat{l}^{p,q}_{\tilde{\mu}} \). This follows from the fact that \( C_g(f) \in l^1_{\tilde{\mu}} \). Similarly, for \( D_g \) it suffices to verify that, if \( c \) is any eventually zero sequence, then \( D_g(c) \in \hat{M}^{p,q}_\mu \). This is true because \( D_g(c) \in M^1_\mu \). \( \Box \)
3. Almost diagonalization of FIOs

For a given function $f$ on $\mathbb{R}^d$ the FIO $T$ with symbol $\sigma$ and phase $\Phi$ can be formally defined by
\begin{equation}
Tf(x) = \int_{\mathbb{R}^d} e^{2\pi i \Phi(x,\eta)} \sigma(x,\eta) \hat{f}(\eta) d\eta.
\end{equation}

To avoid technicalities we take $f \in \mathcal{S}(\mathbb{R}^d)$ or, more generally, $f \in M^1$. If $\sigma \in L^\infty$ and the phase $\Phi$ is real, the integral converges absolutely and defines a function in $L^\infty$.

Assume that the phase function $\Phi(x,\eta)$ fulfills the following properties:
(i) $\Phi \in \mathcal{C}^\infty(\mathbb{R}^{2d})$;
(ii) for $z = (x,\eta)$,
\begin{equation}
|\partial^\alpha \Phi(z)| \leq C_\alpha, \quad |\alpha| \geq 2;
\end{equation}
(iii) there exists $\delta > 0$ such that
\begin{equation}
|\det \partial^2_{x,\eta} \Phi(x,\eta)| \geq \delta.
\end{equation}

If we set
\begin{equation}
\begin{cases}
y = \nabla_\eta \Phi(x,\eta) \\
\xi = \nabla_x \Phi(x,\eta),
\end{cases}
\end{equation}
and solve with respect to $(x,\xi)$, we obtain a mapping $\chi$, defined by $(x,\xi) = \chi(y,\eta)$, which is a smooth bilipschitz canonical transformation. This means that
- $\chi$ is a smooth diffeomorphism on $\mathbb{R}^{2d}$;
- both $\chi$ and $\chi^{-1}$ are Lipschitz continuous;
- $\chi$ preserves the symplectic form, i.e.,
\begin{equation}
dx \wedge d\xi = dy \wedge d\eta.
\end{equation}

Indeed, under the above assumptions, the global inversion function theorem (see e.g. [32]) allows us to solve the first equation in (15) with respect to $x$, and substituting in the second equation yields the smooth map $\chi$. The bounds on the derivatives of $\chi$, which give the Lipschitz continuity, follow from the expression for the derivatives of an inverse function combined with the bounds in (ii) and (iii).

The symplectic nature of the map $\chi$ is classical, see e.g. [7]. Similarly, solving the second equation in (15) with respect to $\eta$ one obtains the function $\chi^{-1}$ with the desired properties.

In this section we prove an almost diagonalization result for FIOs as above, with respect to a Gabor frame. Here we consider the case of regular symbols. In Section 6 we will study the case of symbols in modulation spaces.
Precisely, for a given \(N \in \mathbb{N}\), we consider symbols \(\sigma\) on \(\mathbb{R}^d\) satisfying, for \(z = (x, \eta)\),

\[
|\partial_z^\alpha \sigma(z)| \leq C_\alpha, \quad \text{a.e. } \zeta \in \mathbb{R}^d, \ |\alpha| \leq 2N,
\]

here \(\partial_z^\alpha\) denotes distributional derivatives.

Our goal is to study the decay properties of the matrix of the FIO \(T\) with respect to a Gabor frame. For simplicity, we consider a normalized tight frame \(G\). Consider a phase function satisfying

\[
\Phi(x, \eta) = \sum_{m, n} T_{m, n} \sigma(x, \eta) \eta^n g(x) d\eta
\]

for \(\sigma\) satisfying (16). There exists \(C_N > 0\) such that

\[
|\langle Tg_{m,n}, g_{m',n'} \rangle| \leq C_N |\nabla \Phi(m', n) - (n, m)|^{-2N}.
\]

**Proof.** Recall that the time-frequency shifts interchange under the action of the Fourier transform: \((T_x f)^\wedge = M_{-x} \hat{f}\) and \((M_\eta f)^\wedge = T_\eta \hat{f}\), besides they fulfill the commutation relations \(T_x M_{\eta} = e^{-2\pi i x \eta} M_{\eta} T_x\). Using this properties, we can write

\[
\langle Tg_{m,n}, g_{m',n'} \rangle = \int_{\mathbb{R}^d} Tg_{m,n}(x) M_{n'} T_{m'} \hat{g}(x) dx
\]

\[
= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{2\pi i \Phi(x, \eta)} \sigma(x, \eta) T_n M_{-m} \hat{g}(\eta) M_{-n'} T_{m'} \hat{g}(x) dx d\eta
\]

\[
= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} M_{(0, -m)} T_{(0, -n)} (e^{2\pi i \Phi(x, \eta)} \sigma(x, \eta)) \hat{g}(\eta) M_{-n'} T_{m'} \hat{g}(x) dx d\eta
\]

\[
= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} T_{(-m', 0)} M_{(-n', 0)} M_{(0, -m)} T_{(0, -n)} (e^{2\pi i \Phi(x, \eta)} \sigma(x, \eta)) \hat{g}(x) \hat{g}(\eta) dx d\eta
\]

\[
= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{2\pi i [\Phi(x+m', \eta+n)-(n', m)+(x+m', \eta)]} \sigma(x+m', \eta+n) \hat{g}(x) \hat{g}(\eta) dx d\eta
\]

Since \(\Phi\) is smooth, we expand \(\Phi(x, \eta)\) into a Taylor series around \((m', n)\) and obtain

\[
\Phi(x+m', \eta+n) = \Phi(m', n) + \nabla \Phi(m', n) \cdot (x, \eta) + \Phi_2(m', n)(x, \eta)
\]

where the remainder is given by

\[
\Phi_2(m', n)(x, \eta) = 2 \sum_{|\alpha|=2} \int_0^1 (1-t) \partial^\alpha \Phi((m', n) + t(x, \eta)) dt \frac{(x, \eta)^\alpha}{\alpha!}.
\]

Whence, we can write

\[
|\langle Tg_{m,n}, g_{m',n'} \rangle| = \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{2\pi i [\nabla \Phi(m', n)-(n', m)+(x, \eta)]} e^{2\pi i \Phi_2(m', n)(x, \eta)} \sigma(x+m', \eta+n) \hat{g}(x) \hat{g}(\eta) dx d\eta \right|
\]
For $N \in \mathbb{N}$, using the identity:
\[
(1 - \Delta_z)^N e^{2\pi i \langle \nabla_z \Phi(m', n) - (n', m) \rangle \cdot (x, \eta)} = (2\pi(\nabla_z \Phi(m', n) - (n', m)))^{2N} e^{2\pi i \langle \nabla_z \Phi(m', n) - (n', m) \rangle \cdot (x, \eta)},
\]
we integrate by parts and obtain
\[
\left| \langle T_{g_{m,n}} g_{m',n'} \rangle \right| = \frac{1}{(2\pi(\nabla_z \Phi(m', n) - (n', m)))^{2N}} \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{2\pi i \langle \nabla_z \Phi(m', n) - (n', m) \rangle \cdot (x, \eta)} \times (1 - \Delta_z)^N \left[ e^{2\pi i \Phi_2(m', n, x, \eta)} \sigma(x + m', \eta + n) \bar{g}(x) \bar{g}((\eta) \right] \, dx \, d\eta \right|.
\]
By means of Leibniz’s formula the factor
\[
(1 - \Delta_z)^N \left[ e^{2\pi i \Phi_2(m', n, x, \eta)} \sigma(x + m', \eta + n) \bar{g}(x) \bar{g}((\eta)
\]
can be expressed as
\[
e^{2\pi i \Phi_2(m', n, z)} \sum_{|\alpha| + |\beta| + |\gamma| \leq 2N} C_{\alpha, \beta, \gamma} p(\partial^{|\alpha|} \Phi_2(m', n))(z) (\partial^3 \sigma)(z + (m', n)) \partial^\gamma (\bar{g} \otimes \bar{g})(z),
\]
where $p(\partial^{|\alpha|} \Phi_2(m', n))(z)$ is a polynomial made of derivatives of $\Phi_2(m', n)$ of order at most $|\alpha|$. As a consequence of (ii) we have $\partial_z \Phi_2(m', n, z) = O(z^2)$, which combined with the assumption (16) and the hypothesis $g \in \mathcal{S}(\mathbb{R}^d)$ yields the desired estimate. □

**Remark 3.2.** More generally one can consider symbols satisfying estimates of the form
\[
|\partial^\alpha \sigma(z)| \leq C_\alpha \mu(z), \quad \text{a.e. } z \in \mathbb{R}^d, \quad |\alpha| \leq 2N,
\]
with $\mu \in \mathcal{M}_v$ and also more general windows $g$. Indeed, by arguing as above and using
\[
|\partial^\beta \sigma(z + (m', n))| \leq C_\beta \mu(z + (m', n)) \leq C' C_\beta v(z) \mu(m', n),
\]
one deduces the decay estimates
\[
\left| \langle T_{g_{m,n}} g_{m',n'} \rangle \right| \leq C_N \frac{\mu(m', n)}{(\nabla_z \Phi(m', n) - (n', m))^{2N}},
\]
provided the integral
\[
\int_{\mathbb{R}^d} \sum_{|\alpha| + |\beta| + |\gamma| \leq 2N} C_{\alpha, \beta, \gamma} p(\partial^{|\alpha|} \Phi_2(m', n))(z) v(z) \partial^\gamma (\bar{g} \otimes \bar{g})(z) \, dz,
\]
converges. This is guaranteed if, e.g., $\langle z \rangle^{2N} v(z) \partial^\gamma (\bar{g} \otimes \bar{g})(z) \in L^1$. 

We now assume the additional hypothesis (iii) on the phase, and rewrite (17) in a form convenient for the applications to the continuity of FIOs in the next section. We need the following lemma.

**Lemma 3.1.** Consider a phase function \( \Phi \) satisfying (i), (ii), and (iii). Then

\[
|\nabla_x \Phi(m', n) - n'| + |\nabla_\eta \Phi(m', n) - m| \gtrsim |x(m, n) - m'| + |\xi(m, n) - n'|,
\]

where \((y, \eta) \mapsto (x, \xi)\) is the canonical transformation generated by \(\Phi\).

**Proof.** It suffices to prove the following inequalities:

\[
|\nabla_\eta \Phi(m', n) - m| \gtrsim |x(m, n) - m'|
\]

(22)

\[
|\nabla_x \Phi(m', n) - n'| \geq |\xi(m, n) - n'| - C|\nabla_\eta \Phi(m', n) - m|.
\]

(23)

We observe that, by (15), we have

\[
y = \nabla_\eta \Phi(x(y, \eta), \eta) \quad \forall (y, \eta) \in \mathbb{R}^{2d}
\]

and

\[
\nabla_x \Phi(x, \eta) = \xi(\nabla_\eta \Phi(x, \eta), \eta) \quad \forall (x, \eta) \in \mathbb{R}^{2d}.
\]

(24)

(25)

Hence, we have \(m = \nabla_\eta (x(m, n), n)\), so that

\[
|\nabla_\eta \Phi(m', n) - m| = |\nabla_\eta \Phi(m', n) - \nabla_\eta \Phi(x(m, n), n)|
\]

\[
 \gtrsim |x(m, n) - m'|,
\]

where the last inequality follows from the fact that, for every fixed \(\eta\), the map \(x \mapsto \nabla \Phi_\eta(x, \eta)\) has a Lipschitz inverse, with Lipschitz constant uniform with respect to \(\eta\).

This proves (22).

In order to prove (23) we observe that, in view of (25), it turns out

\[
\nabla_x \Phi(m', n) - n' = \xi(\nabla_\eta \Phi(m', n), n) - n'
\]

\[
= \xi(m + \nabla_\eta \Phi(m', n) - m, n) - n'
\]

\[
= \xi(m, n) - n' + O(\nabla_\eta \Phi(m', n) - m)).
\]

where the last inequality follows from the Taylor formula for the function \(y \mapsto \xi(y, n)\), taking into account that the function \(\xi\) has bounded derivatives.

This proves (23). \(\square\)

Combining the previous lemma with (17) we obtain the following result.

**Theorem 3.3.** Consider a phase function \( \Phi \) satisfying (i), (ii), and (iii), and a symbol satisfying (16). Let \( g \in \mathcal{S}(-d) \). There exists a constant \( C_N > 0 \) such that

\[
|\langle Tg_{m, n}, g_{m', n'} \rangle| \leq C_N \langle \chi(m, n) - (m', n') \rangle^{-2N},
\]

(26)

where \( \chi \) is the canonical transformation generated by \(\Phi\).
This result shows that the matrix representation of a FIO with respect a Gabor frame is well-organized, similarly to the results recently obtained by [6, 27] in terms of shearlets and curvelets frames. More precisely, if \( \sigma \in S_{0,0}^0 \), namely if (16) is satisfied for every \( N \in \mathbb{N} \), then the Gabor matrix of \( T \) is highly concentrated along the graph of \( \chi \).

4. Continuity of FIOs on \( M^p_\mu \)

In this section we study the continuity of FIOs on the modulation spaces \( M^p_\mu \) associated with a weight function \( \mu \in \mathcal{M}_{v_s} \), \( s \geq 0 \). We need the following preliminary lemma.

**Lemma 4.1.** Consider a lattice \( \Lambda \) and an operator \( K \) defined on sequences as

\[
(Kc)_\lambda = \sum_{\nu \in \Lambda} K_{\lambda,\nu} c_\nu,
\]

where

\[
\sup_{\nu \in \Lambda} \sum_{\lambda \in \Lambda} |K_{\lambda,\nu}| < \infty, \quad \sup_{\lambda \in \Lambda} \sum_{\nu \in \Lambda} |K_{\lambda,\nu}| < \infty.
\]

Then \( K \) is continuous on \( l^p(\Lambda) \) for every \( 1 \leq p \leq \infty \) and moreover maps the space \( c_0(\Lambda) \) of sequences vanishing at infinity into itself.

**Proof.** The first part is the classical Schur’s test (see e.g. [23, Lemma 6.2.1]). The second part follows in this way. Since we know that \( K \) is continuous on \( l^\infty \) and the space of eventually zero sequences is dense in \( c_0 \), it suffices to verify that \( K \) maps every eventually zero sequence in \( c_0 \). This follows from the fact that any eventually zero sequence belongs to \( l^1 \) and therefore, since \( K \) is continuous on \( l^1 \), is mapped in \( l^1 \hookrightarrow c_0 \).

We can now state our result.

**Theorem 4.1.** Consider a phase function satisfying (i), (ii), and (iii), and a symbol satisfying (16). Let \( 0 \leq s < 2N - 2d \), and \( \mu \in \mathcal{M}_{v_s} \). For every \( 1 \leq p < \infty \), \( T \) extends to a continuous operator from \( M^p_{\mu \circ \chi \circ v_s} \) into \( M^p_\mu \), and for \( p = \infty \) it extends to a continuous operator from \( \tilde{M}_\mu^\infty \circ v_s \) into \( M_\mu^\infty \).

Recall that \( \tilde{M}_\mu^\infty \) is the closure of \( \mathcal{S}(\mathbb{R}^d) \) in \( M_\mu^\infty \). Moreover, observe that \( \mu \circ \chi \in \mathcal{M}_{v_s} \). Indeed, \( v_s \circ \chi \asymp v_s \), due to the bilipschitz property of \( \chi \).

**Proof.** We first prove that

\[
\|Tf\|_{M^p_{\mu \circ \chi \circ v_s}} \leq C\|f\|_{M^p_\mu},
\]

for every \( f \in \mathcal{S}(\mathbb{R}^d) \). This proves the theorem in the case \( p < \infty \), since \( \mathcal{S}(\mathbb{R}^d) \) is dense in \( M^p_\mu \).
We see at once that, since \( \sigma \in L^\infty \), \( T \) defines a bounded operator from \( M^1 \) into \( L^\infty \). Hence, for all \( f \in S(\mathbb{R}^d) \), we have \( T \infty \) and Theorem 2.2 shows that \( \| f \|_{M^p_{\mu \circ \chi}} \geq \| C_g(f) \|_{\mu \circ \chi} \) and \( \| T \|_{M^p_{\mu \circ \chi}} \leq \| C_g(Tf) \|_{\mu \circ \chi} \). On the other hand, the expansion (10) holds for \( f \) with convergence in \( M^1 \). Therefore
\[
Tf = \sum_{m,n} \langle f, g_{m,n} \rangle Tg_{m,n}
\]
with convergence in \( M^\infty \). Hence,
\[
C_g(Tf)_{m',n'} = \langle Tf, g_{m',n'} \rangle = \sum_{m,n} \langle Tg_{m,n}, g_{m',n'} \rangle \langle f, g_{m,n} \rangle = \sum_{m,n} \langle Tg_{m,n}, g_{m',n'} \rangle C_g(f)_{m,n}.
\]
Therefore we are reduced to proving that the matrix operator
\[
\{c_{m,n}\} \mapsto \sum_{m,n \in \mathbb{Z}^d} \langle Tg_{m,n}, g_{m',n'} \rangle c_{m,n}
\]
is bounded from \( l^p_{\mu \circ \chi} \) into \( l^p_{\mu} \). This follows from Schur’s test (Lemma 4.1) if we prove that, upon setting
\[
K_{m',n',m,n} = \langle Tg_{m,n}, g_{m',n'} \rangle \mu(m',n')/\mu(\chi(m,n)),
\]
we have
\[
K_{m',n',m,n} \in l^\infty_{m,n} l^1_{m',n'},
\]
(28)
and
\[
K_{m',n',m,n} \in l^\infty_{m',n'} l^1_{m,n},
\]
(29)
In view of (26) we have
\[
|K_{m',n',m,n}| \lesssim \langle \chi(m,n) - (m',n') \rangle^{-2N+s} \frac{\mu(m',n')}{\langle \chi(m,n) - (m',n') \rangle^{s} \mu(\chi(m,n))}.
\]
(30)
Now, the last quotient in (30) is bounded because \( \mu \) is \( v_s \)-moderate, so we deduce (28).

Finally, since \( \chi \) is a bilipschitz function we have
\[
|\chi(m,n) - (m',n')| \asymp |(m,n) - \chi^{-1}(m',n')|
\]
so that (29) follows as well.

The case \( p = \infty \) follows analogously by using Theorem 2.3 (with \( p = q = \infty \)), and the last part of the statement of Lemma 4.1.

**Remark 4.2.** Theorem 4.1 with \( v \equiv 1 \) gives, in particular, continuity on the unweighted modulation spaces \( M^p \). If moreover \( p = 2 \), we recapture the classical \( L^2 \)-continuity result by Asada and Fujiwara [1].

Also, Theorem 4.1 applies to \( \mu = v_t \), with \( |t| \leq s \). In that case we obtain continuity on \( M_{v_t} \), because \( v_t \circ \chi \asymp v_t \).
5. Continuity of FIOs on $M^{p,q}$

In this section we study the continuity of FIOs on modulation spaces $M^{p,q}$ possibly with $p \neq q$. As shown in Section 7, under the assumptions of Theorem 4.1 such operators may fail to be bounded when $p \neq q$. The counterexample is given by the phase $\Phi(x, \eta) = x\eta + |x|^2/2$, and symbol $\sigma = 1$, which does not yield a bounded operator on $M^{p,q}$, except for the case $p = q$. Here the obstruction is essentially due to the fact that the map $x \mapsto \nabla_x \Phi(x, \eta)$ has unbounded range. Indeed we will show, for general phases, that if such a map has range of finite diameter, uniformly with respect to $\eta$, then the corresponding operator is bounded on all $M^{p,q}$. To this end we need the following result.

**Proposition 5.1.** Consider an operator defined on sequences on the lattice $\Lambda = \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d$ by

$$(Kc)_{m',n'} = \sum_{m,n} K_{m',n',m,n} c_{m,n}.$$  

(i) If $K \in l_n^{l_n^1} l_m^{l_m^\infty}$, $K$ is continuous on $l_n^{l_n^1} l_m^{l_m^\infty}$.

(ii) If $K \in l_n^{l_n^1} l_m^{l_m^\infty}$, $K$ is continuous on $l_n^{l_n^1} l_m^{l_m^\infty}$.

(iii) If $K \in l_n^{l_n^1} l_m^{l_m^\infty}$, and moreover $K \in l_n^{l_n^1} l_m^{l_m^\infty} \cap l_n^{l_n^1} l_m^{l_m^\infty}$, the operator $K$ is continuous on $l_n^{l_n^1} l_m^{l_m^\infty}$ for every $1 \leq p, q \leq \infty$.

(iv) Assume the hypotheses in (iii). Then $K$ is continuous on all $\tilde{l}_n^{l_n^p} l_m^{l_m^q}$, $1 \leq p, q \leq \infty$.

*Proof.* (i) We have

$$\|Kc\|_{l_n^{l_n^1} l_m^{l_m^\infty}} \leq \sum_{n'} \sup_{m'} \sum_{m,n} |K_{m',n',m,n}| |c_{m,n}|$$

$$\leq \sum_n \left( \sum_{n'} \sup_{m'} \sum_{m} |K_{m',n',m,n}| \right) \sup_m |c_{m,n}|$$

$$\leq \|K\|_{l_n^{l_n^1} l_m^{l_m^\infty}} \|c\|_{l_n^{l_n^1} l_m^{l_m^\infty}}.$$  

(ii) It turns out

$$\|Kc\|_{l_n^{l_n^1} l_m^{l_m^1}} \leq \sum_{n'} \sum_{m'} \left( \sum_{m,n} |K_{m',n',m,n}| |c_{m,n}| \right)$$

$$\leq \sup_{n'} \sum_n \left( \sum_{m'} \sup_{m} \sum_{m} |K_{m',n',m,n}| \sum_{m} |c_{m,n}| \right)$$

$$\leq \|K\|_{l_n^{l_n^1} l_m^{l_m^1}} \|c\|_{l_n^{l_n^1} l_m^{l_m^1}}.$$  

(iii) Since the statement holds for $p = q$ by the classical Schur’s test, and for $(p, q) = (1, \infty)$ and $(p, q) = (\infty, 1)$ by the items (i) and (ii), it follows by complex
interpolation (see (3) on page 128 and (15) on page 134 of [37]) that it holds for all \((p, q)\), except possibly in the cases \(q = \infty, 1 < p < \infty\). For these cases we argue by duality as follows.

In order to prove the continuity of \(K\) on \(l^{\infty}l^p\), it suffices to verify that for any sequences \(c = (c_{m,n}) \in l^{\infty}l^p_m\) and \(d = (d_{m',n'}) \in l^{\infty}l^p_{m'}\), with \(d_{m',n'} \geq 0\), we have

\[
\sum_{m',n'} |(Kc)_{m',n'}|d_{m',n'} \lesssim \|c\|_{l^{\infty}l^p_m} \|d\|_{l^{\infty}l^p_{m'}}. \tag{32}
\]

Now

\[
\sum_{m',n'} |(Kc)_{m',n'}|d_{m',n'} \leq \sum_{m',n'} \sum_{m,n} |K_{m',n',m,n}|c_{m,n}d_{m',n'}
= \sum_{m,n} \left( \sum_{m',n'} |K_{m',n',m,n}|d_{m',n'} \right) |c_{m,n}|
\leq \|c\|_{l^{\infty}l^p} \|\tilde{K}d\|_{l^1l^{p'}}.
\]

where \(\tilde{K}\) is the operator with matrix kernel \(K_{m,n,m',n'} = |K_{m',n',m,n}|\). Since it satisfies the same assumptions as \(K\), it is continuous on \(l^1l^{p'}\), which gives (32).

(iv) Since \(K\) is continuous on \(l^{p,q}\) and by the definition of \(\tilde{l}^{p,q}\), it suffices to verify that \(K\) maps every eventually zero sequence in \(\tilde{l}^{p,q}\). This follows from the fact that \(K\) maps every eventually zero sequence in \(l^1 \hookrightarrow \tilde{l}^{p,q}\), because \(K\) is bounded on \(l^1\).

**Theorem 5.2.** Consider a phase function \(\Phi\) satisfying (i), (ii), and (iii), and a symbol satisfying (16), with \(N > d\). Suppose, in addition, that

\[
\|
\sum_{x,x',\eta} |\nabla_x \Phi(x, \eta) - \nabla_{x'} \Phi(x', \eta)| < \infty.
\]

Then the corresponding Fourier integral operator \(T\) extends to a bounded operator on \(M^{p,q}\) for every \(1 \leq p, q < \infty\) and on \(\tilde{M}^{p,q}\) if \(p = \infty\) or \(q = \infty\).

**Proof.** By arguing as in the proof of Theorem 4.1, it suffices to prove the continuity on \(l^{p,q} = l^{q}_n l^p_m\) if \(p < \infty\) and \(q < \infty\), or \(\tilde{l}^{p,q}\) if \(p = \infty\) or \(q = \infty\), of the operator

\[
\{c_{m,n}\} \mapsto \sum_{m,n \in \mathbb{Z}^d} T_{m',n',m,n} c_{m,n},
\]

where

\[
T_{m',n',m,n} = \langle Tg_{m,n}, g_{m',n'} \rangle.
\]

By applying 5.1, it suffices to verify that

\[
\{T_{m',n',m,n}\} \in l^{\infty}l^1_n l^{\infty}l^1_m,
\]

(34)

\[
\{T_{m',n',m,n}\} \in l^{\infty}l^1_n l^1_m l^{\infty}m',
\]

(35)
because we already see from (26) and (31) that \( \{ T_{m',n',m,n} \} \in l^\infty_{m,n} l^1_{m',n'} \cap l^\infty_{m',n'} l^1_{m,n} \).

Let us now prove (34). It follows from (17) and (23) that
\[
|T_{m',n',m,n}| \lesssim \left( 1 + |\nabla_x \Phi(m', n) - n'|^2 + |\nabla_y \Phi(m', n) - m|^2 \right)^{-N}
\]
\[
\lesssim \left( 1 + |\xi(m, n) - n'|^2 + |\nabla_y \Phi(m', n) - m|^2 \right)^{-N}
\]
\[
\lesssim (1 + |\xi(m, n) - n'|)^{-N} (1 + |\nabla_y \Phi(m', n) - m|)^{-N}.
\]

By (15) we have
\[
\xi(y, \eta) = \nabla_x \Phi(x(y, \eta), \eta), \quad \forall (y, \eta) \in \mathbb{R}^{2d},
\]
so that the hypothesis (33) yields
\[
\xi(m, n) = \xi(0, n) + O(1).
\]

Hence (34) follows.

We now prove (35). As above, it follows from (26) and (22) that
\[
|T_{m',n',m,n}| \lesssim \left( 1 + |\nabla_x \Phi(m', n) - n'|^2 + |\nabla_y \Phi(m', n) - m|^2 \right)^{-N}
\]
\[
\lesssim \left( 1 + |\nabla_x \Phi(m', n) - n'|^2 + |x(m, n) - m'|^2 \right)^{-N}
\]
\[
\lesssim (1 + |\nabla_x \Phi(m', n) - n'|)^{-N} (1 + |x(m, n) - m'|)^{-N}.
\]

By (33) we have
\[
\nabla_x \Phi(m', n) = \nabla_x \Phi(0, n) + O(1),
\]
so that
\[
1 + |\nabla_x \Phi(m', n) - n'| \gtrsim 1 + |\nabla_x \Phi(0, n) - n'|
\]
\[
\gtrsim 1 + |n - \psi(n')|,
\]
where \( \psi \) is the inverse function of the bilipschitz function \( \eta \mapsto \nabla_x \Phi(0, \eta) \). Therefore we obtain (35).

This concludes the proof.

\[\Box\]

**Example 5.3.** Theorem 5.2 applies, in particular, to phases of the type
\[
\Phi(x, \eta) = x \eta + a(x, \eta) \quad \text{where} \quad |\partial_x^\alpha \partial_\eta^\beta a(x, \eta)| \leq C_{\alpha, \beta} \quad \text{for every} \quad 2|\alpha| + |\beta| \geq 2.
\]

In the special case when \( a(x, \eta) = a(\eta) \) is independent of \( x \) and the symbol \( \sigma \equiv 1 \), the FIO reduces to a Fourier multiplier
\[
T f(x) = \int_{\mathbb{R}^d} e^{2\pi i x \eta} e^{2\pi i a(\eta)} \hat{f}(\eta) \, d\eta
\]
and we reobtain the result of [2, Theorem 5] on the continuity of \( T \) on all \( M^{p,q} \), \( 1 \leq p, q \leq \infty \).
6. MODULATION SPACES AS SYMBOL CLASSES

In what follows we shall rephrase the quantity $|\langle T g_{m,n}, g_{m',n'} \rangle|$ in terms of the STFT of the symbol $\sigma$, without assuming the existence of derivatives of $\sigma$. This will be applied to prove the continuity of FIOs with symbols in $M^{\infty,1}$ on modulation spaces $M^p$.

The same arguments as in Theorem 3.1 yield the equality

$$\langle T g_{m,n}, g_{m',n'} \rangle = e^{2\pi i mn} \int_{\mathbb{R}^d} e^{2\pi i \Phi(x,\eta)} \sigma(x,\eta) M_{(-n',-m)} T_{(m',n)} (\hat{g} \otimes \hat{g})(x,\eta) \, dx d\eta.$$ 

Expanding the phase $\Phi$ into a Taylor series around $(m',n)$ we obtain

$$\Phi(x,\eta) = \Phi(m',n) + \nabla_x \Phi(m',n) \cdot (x - m', \eta - n) + T_{(m',n)} \Phi_2(m',n)(x,\eta)$$

where the remainder $\Phi_2(m',n)$ is given by (18).

Inserting this expansion in the integrals above, we can write

$$(37) \quad \langle T g_{m,n}, g_{m',n'} \rangle = e^{2\pi i (mn + \Phi(m',n) - \nabla_x \Phi(m',n) - (m',n))} \int_{\mathbb{R}^d} e^{2\pi i \nabla_x \Phi(m',n)(x,\eta)} \sigma(x,\eta)$$

$$\quad \times M_{(-n',-m)} T_{(m',n)} e^{2\pi i \Phi_2(m',n)(x,\eta)} (\hat{g} \otimes \hat{g})(x,\eta) \, dx d\eta.$$ 

Defining

$$(38) \quad \Psi_{(m',n)}(x,\eta) := e^{2\pi i \Phi_2(m',n)(x,\eta)} (\hat{g} \otimes \hat{g})(x,\eta),$$

and computing the modulus of the left-hand side of (37), we are led to

$$(39) \quad |\langle T g_{m,n}, g_{m',n'} \rangle| = |V_{\Psi_{(m',n)}} \sigma((m',n), (n' - \nabla_x \Phi(m',n), m - \nabla_\eta \Phi(m',n))))|$$

Observe that the window $\Psi_{(m',n)}$ of the STFT above depends on the pair $(m',n)$.

We now study the continuity problem of $T$ when the symbol $\sigma$ is in the modulation space $M^{\infty,1}$.

**Theorem 6.1.** Consider a phase function satisfying (i), (ii), and (iii), and a symbol $\sigma \in M^{\infty,1}$. For every $1 \leq p < \infty$, $T$ extends to a continuous operator on $M^p$, and for $p = \infty$ it extends to a continuous operator on $\dot{M}^\infty$.

By arguing as in the proof of Theorem 4.1, it suffices to prove the continuity on $L^p$ if $1 \leq p < \infty$ and on $L^\infty = c_0$, of the operator (27). In view of Schur’s test (Lemma 4.1) and (39), it suffices to prove the following result.

**Proposition 6.2.** Consider a phase function $\Phi$ satisfying (i) and (ii) and (iii) and a symbol $\sigma \in M^{\infty,1}$. If we set

$$(40) \quad z_{m,n,m',n'} := ((m',n), (n' - \nabla_x \Phi(m',n), m - \nabla_\eta \Phi(m',n))), \quad m, m' \in \alpha \mathbb{Z}^d, \quad n, n' \in \beta \mathbb{Z}^d,$$
then,

\[ \sup_{(m,n) \in \Lambda} \sum_{(m',n') \in \Lambda} |V_{\Psi_{(m',n)}} \sigma(z_{m,n,m',n'})| \lesssim \| \sigma \|_{M^{\infty,1}}. \]

\[ \sup_{(m',n') \in \Lambda} \sum_{(m,n) \in \Lambda} |V_{\Psi_{(m',n)}} \sigma(z_{m,n,m',n'})| \lesssim \| \sigma \|_{M^{\infty,1}}. \]

We need the following lemma.

**Lemma 6.1.** Let \( \Psi_0 \in \mathcal{S}(\mathbb{R}^d) \) with \( \| \Psi_0 \|_{L^2} = 1 \) and \( \Psi_{(m',n)} \) be defined by (38), with \((m',n) \in \Lambda = \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d, \) and \( g \in \mathcal{S}(\mathbb{R}^d). \) Then,

\[ \int_{\mathbb{R}^{4d}} \sup_{(m',n') \in \Lambda} |V_{\Psi_{(m',n)}} \Psi_0(w)| \, dw < \infty. \]

**Proof of Lemma 6.1.** We shall show that

\[ |V_{\Psi_{(m',n)}} \Psi_0(w)| \leq C \langle w \rangle^{-(4d+1)}, \quad \forall (m', n) \in \Lambda. \]

Using the switching property of the STFT:

\[(V_{fg})(x, \eta) = e^{-2\pi i \eta x} (V_g f)(-x, -\eta),\]

we observe that \( |V_{\Psi_{(m',n)}} \Psi_0(w_1, w_2)| = |V_{\Psi_0} \Psi_{(m',n)}(-w_1, -w_2), \) and by the even property of of the weight \( \langle \cdot \rangle, \) relation (44) is equivalent to

\[ |V_{\Psi_0} \Psi_{(m',n)}(w)| \leq C \langle w \rangle^{-(4d+1)}, \quad \forall (m', n) \in \Lambda. \]

Now, the mapping \( V_{\Psi_0} \) is continuous from \( \mathcal{S}(\mathbb{R}^d) \) to \( \mathcal{S}(\mathbb{R}^{4d}) \) (see [23, Chap. 11]). This means that there exists \( M \in \mathbb{N}, \) \( K > 0, \) such that

\[ |V_{\Psi_0} \Psi_{(m',n)}(w)| = |V_{\Psi_0} \Psi_{(m',n)}(w)| \langle w \rangle^{4d+1} \]

\[ \leq \| V_{\Psi_0} \Psi_{(m',n)}(\cdot) \|_{L^\infty(\mathbb{R}^{4d})} \langle w \rangle^{-(4d+1)} \]

\[ \leq K \sum_{|\gamma| + |\delta| \leq M} \| D^\gamma X^\delta \Psi_{(m',n)} \|_{L^\infty(\mathbb{R}^{2d})} \langle w \rangle^{-(4d+1)}, \]

for every \((m', n) \in \Lambda. \) We now claim that \( \Psi_{(m',n)} \in \mathcal{S}(\mathbb{R}^{2d}) \) uniformly with respect to \( (m', n). \) This is proved as follows: the function \( e^{2\pi i \Phi_2(x, \eta)} \) is in \( C^\infty(\mathbb{R}^{2d}) \) and possesses derivatives dominated by powers \( \langle x, \eta \rangle^k, k \in \mathbb{N}, \) uniformly with respect to \( (m', n), \) due to (13); since \( (\hat{g} \otimes \hat{g}) \in \mathcal{S}(\mathbb{R}^{2d}), \) it follows that \( \Psi_{(m',n)} \in \mathcal{S}(\mathbb{R}^{2d}), \) with semi-norms uniformly bounded:

\[ p_M(\Psi_{(m',n)}) := \sum_{|\gamma| + |\delta| \leq M} \| D^\gamma X^\delta \Psi_{(m',n)} \|_{L^\infty} \leq C, \quad \forall (m', n) \in \Lambda. \]
Consequently,
\[ |V_{\psi_0}\Psi_{(m',n)}(w)| \leq K \sum_{|\gamma|+|\delta| \leq M} \|D^\gamma X^\delta \Psi_{(m',n)}\|_{L^\infty(\mathbb{R}^{2d})} \langle w \rangle^{-(4d+1)} \leq KC_M \langle w \rangle^{-(4d+1)}, \]
for every \((m', n) \in \Lambda\), as desired.

**Proof of Proposition 6.2.** We shall prove (41). First, Lemma 2.1 for \(g_1 = \gamma = \Psi_0\), yields
\[ |V_{\psi_{(m',n)}} \sigma(z)| \leq \langle |V_{\psi_0} \sigma| \ast |V_{\psi_{(m',n)}} \Psi_0| \rangle(z), \quad z \in \mathbb{R}^{4d}, \]
so that
\[
\sum_{(m', n') \in \Lambda} |V_{\psi_{(m',n)}} \sigma(z_{m,n,m',n'})|
\leq \int_{\mathbb{R}^{4d}} \sum_{(m', n') \in \Lambda} |V_{\psi_0} \sigma(z_{m,n,m',n'} - w)| \langle V_{\psi_{(m',n)}} \Psi_0 \rangle(w) \, dw
\leq \int_{\mathbb{R}^{4d}} \sum_{(m', n') \in \Lambda} |V_{\psi_0} \sigma(z_{m,n,m',n'} - w)| \sup_{(m', n') \in \Lambda} \langle V_{\psi_{(m',n)}} \Psi_0 \rangle(w) \, dw
\leq \sup_{w \in \mathbb{R}^{4d}} \sum_{(m', n') \in \Lambda} |V_{\psi_0} \sigma(z_{m,n,m',n'} - w)| \int_{\mathbb{R}^{4d}} \sup_{(m', n') \in \Lambda} \langle V_{\psi_{(m',n)}} \Psi_0 \rangle(w) \, dw
\leq C \sup_{w \in \mathbb{R}^{4d}} \sum_{(m', n') \in \Lambda} |V_{\psi_0} \sigma(z_{m,n,m',n'} - w)|,
\]
where the last majorization is due to Lemma 6.1. Since,
\[
\sum_{(m', n') \in \Lambda} |V_{\psi_0} \sigma(z_{m,n,m',n'} - w)| \leq \sum_{(m', n') \in \Lambda} \sup_{u_1 \in \mathbb{R}^{2d}} |V_{\psi_0} \sigma(u_1, \tilde{z}_{m,n,m',n',w_2})|,
\]
with
\[
\tilde{z}_{m,n,m',n',w_2} := (n' - \nabla_\eta \Phi(m', n), m - \nabla_\eta \Phi(m', n)) - w_1, \quad w = (w_1, w_2) \in \mathbb{R}^{2d},
\]
we shall prove that
\[ \sum_{(m', n') \in \Lambda} \sup_{u_1 \in \mathbb{R}^{2d}} |V_{\psi_0} \sigma(u_1, \tilde{z}_{m,n,m',n',w_2})| \lesssim \|\sigma\|_{M^{\infty,1}}, \]
uniformly with respect to \((m, n) \in \Lambda, w_2 \in \mathbb{R}^{2d}\). For every fixed \((m, n)\), the set \(\mathcal{X} = \mathcal{X}_{m,n,w_2}\), given by
\[ \mathcal{X}_{m,n,w_2} = \{ \tilde{z}_{m,n,m',n',w_2} \mid (m', n') \in \Lambda, w_2 \in \mathbb{R}^{2d} \}, \]
is separated, uniformly with respect to \((m, n), w_2\). Indeed, given \((m_1', n_1') \neq (m_2', n_2')\), if \(m_1' \neq m_2'\),
\[
|\tilde{z}_{m,n,m',n',w_2} - \tilde{z}_{m,n,m'_2,n'_2,w_2}| \geq |\nabla_\eta \Phi(m_1', n) - \nabla_\eta \Phi(m_2', n)| \geq C|m_1' - m_2'| \geq \alpha C,
\]
uniformly with respect to \((m, n), w_2\), because the mapping \(x \mapsto \nabla_\eta \Phi(x, \eta)\) has an inverse that is Lipschitz continuous, thanks to (13) and (14). On the other hand, if \(m'_1 = m'_2\),

\[
|\tilde{z}_{m,n,m'_1,n'_1,w_2} - \tilde{z}_{m,n,m'_2,n'_2,w_2}| \geq |n'_1 - n'_2| \geq \beta, \quad \forall (m, n) \in \Lambda, w_2 \in \mathbb{R}^{2d}.
\]

Hence, \(\mathcal{X}\) is separated uniformly with respect to \((m, n), w_2\). Now, we apply Proposition 2.1 (with \(p = q = 1\)) to the function

\[
F(u_2) = \sup_{u_1 \in \mathbb{R}^{2d}} |V_{\psi_0} \sigma(u_1, u_2)|, \quad u_2 \in \mathbb{R}^{2d},
\]

which is lower semi-continuous, being \(V_{\psi_0} \sigma\) continuous. We obtain

\[
(47) \quad \|F|_{L^1} \leq C \|F\|_{W(L^\infty, L^1)} = \|V_{\psi_0} \sigma\|_{W(C, L^{\infty, 1})}.
\]

If the symbol \(\sigma\) is in \(M^{\infty, 1}\), by Lemma 2.2 the STFT \(V_{\psi_0} \sigma\) belongs to the Wiener amalgam space \(W(\mathcal{F}L^1, L^{\infty, 1})\), and

\[
\|V_{\psi_0} \sigma\|_{W(C, L^{\infty, 1})} \lesssim \|V_{\psi_0} \sigma\|_{W(\mathcal{F}L^1, L^{\infty, 1})} \lesssim \|\sigma\|_{M^{\infty, 1}} \|\psi_0\|_{M^1}.
\]

The first inequality is due to \(\mathcal{F}L^1 \hookrightarrow \mathcal{C}\) and the inclusion relations between Wiener amalgam spaces. Combining this inequality with (47) we obtain (46), uniformly with respect to \((m, n)\) and \(w_2\), that is (41).

The estimate (42) is obtained by similar arguments. \(\square\)

**Remark 6.3.** We observe that the continuity on \(M^2 = L^2\) of FIOs as above, with symbols in \(M^{\infty, 1}\), was already proved in [5] by other methods.

**7. The Case of Quadratic Phases: Metaplectic Operators**

In this section we briefly discuss the particular case of quadratic phases, namely phases of the type

\[
(48) \quad \Phi(x, \eta) = \frac{1}{2} Ax \cdot x + Bx \cdot \eta + \frac{1}{2} C\eta \cdot \eta + \eta_0 \cdot x - x_0 \cdot \eta,
\]

where \(x_0, \eta_0 \in \mathbb{R}^d\), \(A, C\) are real symmetric \(d \times d\) matrices and \(B\) is a real \(d \times d\) nondegenerate matrix.

It is easy to see that, if we take the symbol \(\sigma \equiv 1\) and the phase (48), the corresponding FIO \(T\) is (up to a constant factor) a metaplectic operator. This can be seen by means of the easily verified factorization

\[
(49) \quad T = M_{x_0} U_A D_B \mathcal{F}^{-1} U_C \mathcal{F} T_{x_0},
\]

where \(U_A\) and \(U_C\) are the multiplication operators by \(e^{\pi i Ax \cdot x}\) and \(e^{\pi i C\eta \cdot \eta}\) respectively, and \(D_B\) is the dilation operator \(f \mapsto f(B \cdot)\). Each of the factors is (up to a constant factor) a metaplectic operator (see e.g. the proof of [31, Theorem 18.5.9]), so \(T\) is.
The corresponding canonical map, defined by (15), is now an affine symplectic map. For the benefit of the reader, some important special cases are detailed in the table below.

<table>
<thead>
<tr>
<th>operator</th>
<th>phase $\Phi(x, \eta)$</th>
<th>canonical transformation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_{x_0}$</td>
<td>$(x - x_0) \cdot \eta$</td>
<td>$\chi(y, \eta) = (y + x_0, \eta)$</td>
</tr>
<tr>
<td>$M_{\eta_0}$</td>
<td>$(\eta + \eta_0) \cdot x$</td>
<td>$\chi(y, \eta) = (y, \eta + \eta_0)$</td>
</tr>
<tr>
<td>$D_B$</td>
<td>$Bx \cdot \eta$</td>
<td>$\chi(y, \eta) = (B^{-1}y, B\eta)$</td>
</tr>
<tr>
<td>$U_A$</td>
<td>$x \cdot \eta + \frac{1}{2}Ax \cdot x$</td>
<td>$\chi(y, \eta) = (y, \eta + Ax)$</td>
</tr>
</tbody>
</table>

However one should observe that there are metaplectic operators, as the Fourier transform, which cannot be expressed as FIOs of the type (12).

Metaplectic operators are known to be bounded on $M_{v_s}^p$, see e.g. [23, Proposition 12.1.3]. This also follows from Theorem 4.1. Indeed, since $\chi$ is a bilipschitz function, we have $v_s \circ \chi \asymp v_s$.

Also, Theorem 5.2 applies to quadratic phases whose affine symplectic map $\chi$ is (up to translations on the phase space) defined by an upper-triangular matrix, which happens precisely when $A = 0$. Indeed, we obtain the map $\chi$ by solving

$$\begin{align*}
y &= Bx + C\eta - x_0 \\
\xi &= Ax + B\eta + \eta_0.
\end{align*}$$

The phase condition (14) here becomes

$$\det B \neq 0,$$

so that $B$ is an invertible matrix and $x = B^{-1}y - B^{-1}C\eta + B^{-1}x_0$. Whence, the mapping $\chi : (y, \eta) \longmapsto (x, \xi)$ is given by

$$\begin{bmatrix} x \\ \xi \end{bmatrix} = \begin{bmatrix} B^{-1} & -B^{-1}C \\ AB^{-1} & B - AB^{-1}C \end{bmatrix} \begin{bmatrix} y \\ \eta \end{bmatrix} + \begin{bmatrix} B^{-1}x_0 \\ AB^{-1}x_0 + \eta_0 \end{bmatrix}.$$

When $A = 0$ the phase $\Phi$ satisfies (33) and, consequently, the corresponding operators are bounded on all $M^{p,q}$. This can also be verified by means of the factorization (49) (with $A = 0$). Indeed the continuity of the operators $M_{\eta_0}$, $T_{x_0}$ and $D_B$ is easily seen, whereas that of the Fourier multiplier $\mathcal{F}^{-1}U_C\mathcal{F}$ was shown, e.g., in [25, Lemma 2.1].

On the other hand, generally the metaplectic operators are not bounded on $M^{p,q}$ if $p \neq q$. An example is given by the Fourier transform itself (see [14]). An example which instead falls in the class of FIOs considered here is the following one.

**Proposition 7.1.** The multiplication $U_{1_d}$ is unbounded on $M^{p,q}$, for every $1 \leq p, q \leq \infty$, with $p \neq q$. 
Proof. We have $U_t f(x) = e^{\pi i|x|^2} f(x)$. For $\lambda > 0$, we consider the one-parameter family of functions $f(x) = e^{-\pi x^2} \in M^{p,q}$, so that $\hat{f}(\eta) = \lambda^{-d/2} e^{-\pi(1/\lambda)|\eta|^2}$. For every $1 \leq p, q \leq \infty$, by [11, Lemma 5.3], we have

$$\| f \|_{M^{p,q}} = \| \hat{f} \|_{W(FL^p, L^q)} \lesssim \frac{(\lambda + 1)^{d(\frac{1}{2} - \frac{1}{q})}}{\lambda^{\frac{d}{2q}}(\lambda^2 + \lambda)^{\frac{d}{2}(\frac{1}{p} - \frac{1}{q})}}.$$ 

Since $U f(x) e^{-\pi(\lambda - i)|x|^2}$, so that $\hat{U} f(\eta) = (\lambda - i)^{-d/2} e^{-\pi(1/(\lambda - i))|\eta|^2}$, the same formula as above yields

$$\| U f \|_{M^{p,q}} \lesssim \frac{[(\lambda + 1)^2 + 1]^{d(\frac{1}{2} - \frac{1}{q})}}{\lambda^{\frac{d}{2q}}(\lambda^2 + \lambda + 1)^{\frac{d}{2}(\frac{1}{p} - \frac{1}{q})}}.$$ 

As $\lambda \to 0$, we have

$$\| U f \|_{M^{p,q}} \lesssim \lambda^{-\frac{d}{2q}}, \quad \| f \|_{M^{p,q}} \lesssim \lambda^{-d/2 - \frac{d}{2}(\frac{1}{p} - \frac{1}{q})}$$

so that, if we assume $\| U f \|_{M^{p,q}} \leq C \| f \|_{M^{p,q}}$, then $1/p - 1/q \geq 0$, that is $p \leq q$.

Moreover, the same argument applies to the adjoint operator $U^* f(x) = e^{-\pi i|x|^2} f(x)$.

Now we show that $p = q$. By contradiction, if $U$ were bounded on $M^{p,q}$, with $p < q$, its adjoint $U^*$ would satisfy

$$\| U^* f \|_{M^{p',q'}} \leq C \| f \|_{M^{p',q'}} \quad \forall f \in \mathcal{S}(\mathbb{R}^d),$$

with $q' < p'$, which is a contradiction to what just proved. \qed

References


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