Localization operators and exponential weights for modulation spaces

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Abstract. We study localization operators within the framework of ultra-distributions. More precisely, given a symbol $a$ and two windows $\varphi_1, \varphi_2$, we investigate the multilinear mapping from $(a, \varphi_1, \varphi_2) \in \mathcal{S}^{(1)'}(\mathbb{R}^{2d}) \times \mathcal{S}^{(1)'}(\mathbb{R}^{d}) \times \mathcal{S}^{(1)'}(\mathbb{R}^{d})$ to the localization operator $A_{\varphi_1, \varphi_2}a$. Results are formulated in terms of modulation spaces with weights which may have exponential growth. We give sufficient and necessary conditions for $A_{\varphi_1, \varphi_2}a$ to be bounded or to belong to a Schatten class. As an application, we study symbols defined by ultra-distributions with compact support, that give trace class localization operators.

1. Introduction

Time-frequency localization operators are pseudo-differential operators $A_{\varphi_1, \varphi_2}a$, where $a$ is the symbol of the operator and $\varphi_1, \varphi_2$ are the analysis and synthesis windows, respectively (see below for an explicit expression). In the literature they are also known as Anti-Wick operators, Gabor multipliers, Toeplitz operators or wave packets. The terminology Time-frequency localization operators is due to Daubechies [8], who used them as a mathematical tool to localize a signal in the time-frequency plane. We recall that the case $\varphi_1, \varphi_2 = e^{-\pi \|x\|^2}$ comes back to Berezin [1]; in the latter framework the operator $A_{\varphi_1, \varphi_2}a$ comes as the result of a quantization procedure and is used in the PDE context, see Shubin [22]. From this point of view, it is natural to compare localization operators with classical pseudodifferential operators, as defined originally by Hörmander, Kohn and Nirenberg, and generalized and used by many authors, see Hörmander [16]. If we consider in particular $L_{\sigma}$, the Weyl pseudodifferential operator with symbol $\sigma$, a precise connection is obtained by observing that $A_{\varphi_1, \varphi_2}a = L_{\sigma}$ if

\begin{equation}
\sigma = a \ast W(\varphi_2, \varphi_1),
\end{equation}

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where \( W(\varphi_2, \varphi_1) \) is the cross-Wigner distribution defined below (see (3.2)). It follows from (1.1) that, if \( \varphi_1, \varphi_2 \) are very smooth, for instance if they belong to the Schwartz space \( \mathcal{S} \), then even very rough symbols \( a \) for the localization operator give rise to smooth Weyl symbols \( \sigma \). So, with respect to the classical pseudodifferential calculus, we may allow singular symbols for localization operators and nevertheless obtain good properties, in particular \( L^2 \)-boundedness. This was the basic idea in [6], where continuity and Schatten properties were investigated in the frame of \( \mathcal{S}' \), the space of tempered distributions. In particular, in [6] the interplay between the roughness of the symbol \( a \) and the time-frequency concentration of the windows \( \varphi_1, \varphi_2 \) was studied in detail, by using modulation spaces \( M_{p,q}^{r,s} \) of Feichtinger [9].

As a striking example, it was observed in [6] that any Schwartz distribution with compact support \( a \in \mathcal{E}' \) gives a trace class operator \( A_{\varphi_1,\varphi_2}^a \), if \( \varphi_1, \varphi_2 \in \mathcal{S} \). In the present paper we carry the results of [6] to, say, their extreme consequences. Window functions are assumed to belong to ultra-modulation spaces, cf. [15, 19, 24]; namely, we introduce \( M_{1,w}^{r,s} \) with \( w \) exponential weight depending on \( s \), \( 0 \leq s < \infty \) (the limit case \( s = 0 \) gives the unweighted space), having as projective and inductive limit the Gelfand-Shilov space \( \mathcal{S}^{(1)} \), respectively, subspaces of the Schwartz set \( \mathcal{S} \). Then, we may allow symbols \( a \) which are ultra-distributions in \( \mathcal{S}' \), and find sufficient and necessary conditions for the \( L^2 \)-boundedness. In this way we extend [6], obtaining in particular that for windows \( \varphi_1, \varphi_2 \in \mathcal{S}^{(1)} \), every ultra-distribution with compact support \( a \in \mathcal{E}' , t > 1 \) (see [20, 21]), defines a trace class operator \( A_{\varphi_1,\varphi_2}^a \). For example, we may consider an infinite sum of derivatives of the \( \delta \) distribution at the origin:

\[
a = \sum_{\mu \in \mathbb{N}_d^2} \frac{1}{(\mu!)^r} \partial^\mu \delta,
\]

which belongs to \( \mathcal{E}' \), if \( r > t \).

To state our main result more precisely, let us recall some definitions. First, recall that Gelfand-Shilov type space \( \mathcal{S}^{(1)} \) can be defined as a class of analytic functions \( f \) such that for every \( h, k > 0 \),

\[
\| f(x) e^{k|x|} \|_{L^\infty} < \infty \quad \text{and} \quad \| \hat{f}(\omega) e^{k|\omega|} \|_{L^\infty} < \infty.
\]

Its strong dual is a space of tempered ultra-distributions of Beurling type, \( \mathcal{S}'^{(1)} \). The introduction of modulation spaces requires some time-frequency tools. Translation and modulation operators are given by

\[
T_x f(t) = f(t - x) \quad \text{and} \quad M_\omega f(t) = e^{2\pi i \omega t} f(t).
\]

For a fixed non-zero \( g \in \mathcal{S}^{(1)}(\mathbb{R}^d) \) the short-time Fourier transform (STFT) of \( f \in \mathcal{S}(\mathbb{R}^d) \) with respect to the window \( g \) is given by

\[
V_g f(x, \omega) = \langle f, M_\omega T_x g \rangle = \int_{\mathbb{R}^d} f(t) \overline{g(t - x)} e^{-2\pi i \omega t} dt,
\]
which can be extended to \( f \in \mathcal{S}^{(1)'}(\mathbb{R}^d) \) by duality. For a fixed \( g \in \mathcal{S}^{(1)}(\mathbb{R}^d) \) the following characterization of \( \mathcal{S}^{(1)} \) holds:

\[
f \in \mathcal{S}^{(1)}(\mathbb{R}^d) \Leftrightarrow V_g f \in \mathcal{S}^{(1)}(\mathbb{R}^{2d}).
\]

For \( z = (x, \omega) \in \mathbb{R}^{2d} \), and \( s > 0 \), we consider the exponential weights \( w_s(z) := e^{s\|z\|_2} \). Note that \( w_s \) belongs to \( \mathcal{S}^{(1)'} \) but it is not a tempered distribution. For \( f \in \mathcal{S}^{(1)'}(\mathbb{R}^d) \), we define the modulation space \( M^1_{s,w_s}(\mathbb{R}^d) \) by the norm

\[
\|f\|_{M^1_{s,w_s}} := \|(V_g f) w_s\|_{L^1(\mathbb{R}^{2d})} < \infty
\]

for some (hence all) non-zero \( g \in \mathcal{S}^{(1)}(\mathbb{R}^d) \). Its dual space \( M^\infty_{1/w_s}(\mathbb{R}^d) \) possesses the norm

\[
\|f\|_{M^\infty_{1/w_s}} := \|(V_g f)^{1/w_s}\|_{L^\infty(\mathbb{R}^{2d})} < \infty.
\]

It can be shown that \( \mathcal{S}^{(1)} \) can be represented as projective limit of weighted modulation spaces, see Proposition 2.3. This implies that \( \mathcal{S}^{(1)'} \) can be viewed as the union of the corresponding duals. Namely, we have

\[
\mathcal{S}^{(1)} = \bigcap_{s \geq 0} M^1_{s,w_s} \quad \text{and} \quad \mathcal{S}^{(1)'} = \bigcup_{s \geq 0} M^\infty_{1/w_s}.
\]

The time-frequency localization operator \( A^{\varphi_1, \varphi_2}_a \) with symbol \( a \) and windows \( \varphi_1, \varphi_2 \) is defined to be

\[
A^{\varphi_1, \varphi_2}_a f(t) = \int_{\mathbb{R}^{2d}} a(x, \omega) V_{\varphi_1} f(x, \omega) M_x T_{\omega} \varphi_2(t) \, dx \, d\omega
\]

or, in the weak sense,

\[
\langle A^{\varphi_1, \varphi_2}_a f, g \rangle = \langle a V_{\varphi_1} f, V_{\varphi_2} g \rangle = \langle a, V_{\varphi_1}^* V_{\varphi_2} g \rangle, \quad f, g \in \mathcal{S}^{(1)}(\mathbb{R}^d).
\]

By (1.5) if \( a \in \mathcal{S}^{(1)'}(\mathbb{R}^{2d}) \) and \( \varphi_1, \varphi_2 \in \mathcal{S}^{(1)}(\mathbb{R}^d) \), then (1.4) is a well-defined continuous operator from \( \mathcal{S}^{(1)}(\mathbb{R}^d) \) to \( \mathcal{S}^{(1)'}(\mathbb{R}^d) \).

Our results for the boundedness of a localization operator can be simplified as follows.

**Theorem 1.1.** If \( a \in M^\infty_{1/w_s}(\mathbb{R}^{2d}) \), and \( \varphi_1, \varphi_2 \in M^1_{s,w_s}(\mathbb{R}^d) \), then \( A^{\varphi_1, \varphi_2}_a \) is bounded on \( L^2(\mathbb{R}^d) \) (and on each modulation space as well), with operator norm at most

\[
\|A^{\varphi_1, \varphi_2}_a\|_{op} \leq C \|a\|_{M^\infty_{1/w_s}} \|\varphi_1\|_{M^1_{s,w_s}} \|\varphi_2\|_{M^1_{s,w_s}}.
\]

Section 2 is devoted to a more detailed study of the involved function spaces. Novelty here is the extension to exponential weights of the results of [19, 23, 24] for the sub-exponential case. In Section 3 we prove Theorem 1.1 and give more general results. Parts of the proofs are only sketched, because they are easy modifications of [6]. In Section 4 we prove that ultra-distributions with compact support as symbols give trace class operators.

Let us finally observe that our results apply in particular to the anti-Wick case, i.e. localization operators with Gaussian windows: \( \varphi_1(t) = \varphi_2(t) = e^{-s\|t\|^2} \). Actually, this choice of the windows suggests that we could take as symbols even
more general ultra-distributions, in the duals of the Gelfand-Shilov class \( S^{(\alpha)} \), with \( 1/2 \leq \alpha < 1 \). Nevertheless, the weights of the corresponding modulation spaces should allow a growth at infinity of the type \( e^{\|x\|^c} \), \( c > 0 \) and \( 1 < r \leq 2 \), losing the sub-multiplicative property, which is essential in our arguments. To be definite, we set the notation we shall use (and have already used) in this paper.

**Notation.** We define \( xy = x \cdot y \), the scalar product on \( \mathbb{R}^d \). Given a vector \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \), the partial derivative with respect to \( x_j \) is denoted by \( \partial_j = \frac{\partial}{\partial x_j} \). Given a multi-index \( p = (p_1, \ldots, p_d) \geq 0 \), i.e., \( p \in \mathbb{N}^d_0 \) and \( p_j \geq 0 \), we write \( \partial^p = \partial_1^{p_1} \cdots \partial_d^{p_d} \); moreover, we write \( x^p = (x_1, \ldots, x_d)^{(p_1, \ldots, p_d)} = \prod_{i=1}^d x_i^{p_i} \). We shall denote the Euclidean norm by \( \|x\| \), the \( \ell_1 \)-norm by \( \|x\|_1 = \sum_{i=1}^d |x_i| \) and the \( \ell_\infty \)-norm by \( \|x\|_\infty = \max\{|x_1|, \ldots, |x_d|\} \); the absolute value of a vector \( x \) means the vector of absolute values \( |x| = (|x_1|, \ldots, |x_d|) \). We write \( h|x|^{1/\alpha} = \sum_{i=1}^d h_i |x_i|^{1/\alpha_i} \). Moreover, for \( p \in \mathbb{N}^d_0 \) and \( \alpha \in \mathbb{R}^d_+ \), we set \( (p!)^\alpha = (p_1!)^{\alpha_1} \cdots (p_d!)^{\alpha_d} \), while as standard \( p! = p_1! \cdots p_d! \). In the sequel, a real number \( r \in \mathbb{R}_+ \) may play the role of the vector with constant components \( r_j = r \), so for \( \alpha \in \mathbb{R}^d_+ \), by writing \( \alpha > r \) we mean \( \alpha_j > r \) for all \( j = 1, \ldots, d \).

The Schwartz class is denoted by \( S(\mathbb{R}^d) \), the space of tempered distributions by \( S'(\mathbb{R}^d) \). We use the brackets \( \langle f, g \rangle \) to denote the extension to \( S^{(1)}(\mathbb{R}^d) \times S^{(1)}(\mathbb{R}^d) \) of the inner product \( \langle f, g \rangle = \int f(t) \overline{g(t)} dt \) on \( L^2(\mathbb{R}^d) \). The space of smooth functions with compact support on \( \mathbb{R}^d \) is denoted by \( D(\mathbb{R}^d) \). The Fourier transform is normalized to be \( f(\omega) = \mathcal{F}(f)(\omega) = \int f(t) e^{-2\pi i t \omega} dt \).

We denote by \( \langle \cdot \rangle^s \) the polynomial weights
\[
\langle (x, \omega)^s \rangle = \langle z \rangle^s = (1 + x^2 + \omega^2)^{s/2}, \quad z = (x, \omega) \in \mathbb{R}^{2d}, \quad s \in \mathbb{R}
\]
and the product weights \( \langle \cdot \rangle^s \otimes \langle \cdot \rangle^s \) signifying \( \langle x \rangle^s(\omega)^s \).

The singular values \( \{ s_k(L) \}_{k=1}^\infty \) of a compact operator \( L \in B(L^2(\mathbb{R}^d)) \) are the eigenvalues of the positive self-adjoint operator \( \sqrt{L^2} \). For \( 1 \leq p < \infty \), the Schatten class \( S_p \) is the space of all compact operators whose singular values lie in \( l^p \). For consistency, we define \( S_\infty := B(L^2(\mathbb{R}^d)) \) to be the space of bounded operators on \( L^2(\mathbb{R}^d) \). In particular, \( S_2 \) is the space of Hilbert-Schmidt operators, and \( S_1 \) is the space of trace class operators.

Throughout the paper, we shall use the notation \( A \lesssim B \) to indicate \( A \leq cB \) for a suitable constant \( c > 0 \), whereas \( A \asymp B \) means that \( c^{-1}A \leq B \leq cA \) for some \( c \geq 1 \). The symbol \( B_1 \hookrightarrow B_2 \) denotes the continuous and dense embedding of the topological vector space \( B_1 \) into \( B_2 \).

## 2. Function spaces

### 2.1. Gelfand-Shilov Type Spaces

We introduce the definition and properties of Gelfand-Shilov spaces which will be used in the sequel. For more details we refer the reader to [11, 15, 18, 23].
**Definition 2.1.** Let \( \alpha, \beta \in \mathbb{R}^d \), and assume \( A_1, \ldots, A_d, B_1, \ldots, B_d > 0 \). Then, the Gelfand-Shilov type space \( S^{\alpha,B}_{\beta,A} = S^{\alpha,B}_{\beta,A}(\mathbb{R}^d) \) is defined by

\[
S^{\alpha,B}_{\beta,A} = \{ f \in C^\infty(\mathbb{R}^d) \mid (\exists C > 0) \| x^p \partial^q f \|_{L^\infty} \leq CA^p (p!)^\alpha B^q (q!)^\beta, \ \forall p, q \in \mathbb{N}_0^d \}.
\]

Their projective and inductive limits are denoted by

\[
\Sigma^\beta_\alpha = \text{proj} \lim_{A > 0, B > 0} S^{\alpha,B}_{\beta,A}; \quad \Sigma^\beta_\alpha := \text{ind} \lim_{A > 0, B > 0} S^{\alpha,B}_{\beta,A}.
\]

The above spaces are contained in \( S \), and moreover for any \( \alpha, \beta \geq 0 \) the Fourier transform is a topological isomorphism between \( S^\alpha_\beta \) and \( S^\beta_\alpha \) (\( \mathcal{F}(S^\alpha_\beta) = S^\beta_\alpha \)) and extends to a continuous linear transform from \( (S^\alpha_\beta)' \) onto \( (S^\beta_\alpha)' \). In particular, if \( \alpha = \beta \) and \( \alpha \geq 1/2 \) then \( \mathcal{F}(S^\alpha_\alpha) = S^\alpha_\alpha \). Similar assertions hold for \( \Sigma^\beta_\alpha \). Due to this fact, corresponding dual spaces are referred to as tempered ultra-distributions (of Beurling or Roumieu type).

Note that \( S^\beta_\alpha \) is nontrivial if and only if \( \alpha + \beta > 1 \), or \( \alpha + \beta = 1 \) and \( \alpha \beta > 0 \). We will use the notation \( S^{(\alpha)} = \Sigma^\alpha_0, S^{(\alpha)} = S^\alpha_\alpha \). We have \( S^\beta_\alpha \hookrightarrow S^\beta_\beta \hookrightarrow S \). The following result is a characterization of Gelfand-Shilov type spaces, see [3, 11, 15, 17].

**Theorem 2.2.** Let \( \alpha, \beta \in \mathbb{R}^d \), \( \alpha, \beta \geq 1/2 \), i.e., \( \alpha_i, \beta_i \geq 1/2 \), for every \( i = 1, \ldots, d \). The following conditions are equivalent:

a) \( f \in S^\alpha_\beta \) (resp. \( f \in \Sigma^\alpha_\beta \)).

b) There exist (resp. for every) constants \( A_1, \ldots, A_d, B_1, \ldots, B_d > 0 \)

\[
\| x^p f \|_{L^\infty} \leq CA^p (p!)^\beta \quad \text{and} \quad \| \omega^q \hat{f} \|_{L^\infty} \leq CB^q (q!)^\alpha, \quad \forall p, q \in \mathbb{N}_0^d,
\]

for some \( C > 0 \).

c) There exist (resp. for every) constants \( A_1, \ldots, A_d, B_1, \ldots, B_d > 0 \)

\[
\| x^p f \|_{L^\infty} \leq CA^p (p!)^\beta \quad \text{and} \quad \| \partial^q f \|_{L^\infty} \leq CB^q (q!)^\alpha, \quad \forall p, q \in \mathbb{N}_0^d,
\]

for some \( C > 0 \).

d) There exist (resp. for every) constants \( h_1, \ldots, h_d, k_1, \ldots, k_d > 0 \)

\[
\| f e^{h_j |x|^1/\alpha} \|_{L^\infty} < \infty \quad \text{and} \quad \| f e^{k_j |x|^1/\beta} \|_{L^\infty} < \infty.
\]

(2.1)

e) There exist (resp. for every) constants \( h_1, \ldots, h_d, B_1, \ldots, B_d > 0 \)

\[
\| (\partial^q f) e^{h_j |x|^1/\alpha} \|_{L^\infty} < CB^q (q!)^\alpha,
\]

for some \( C > 0 \).

(2.2)

Gelfand-Shilov type spaces enjoy the embedding property below.

**Lemma 2.1.** For every \( \alpha_1 < \alpha_2 \) and \( \beta_1 < \beta_2 \) we have

\[
S^{\alpha_1}_{\beta_1} \hookrightarrow S^{\alpha_2}_{\beta_2}.
\]

(2.3)
Lemma 2.2. Let \( g \in \mathcal{S}^{(1)}(\mathbb{R}^d) \setminus \{0\} \). Then for \( f \in (\mathcal{S}^{(1)})'(\mathbb{R}^d) \), the following characterization holds:
\[
(2.4) \quad f \in \mathcal{S}^{(1)}(\mathbb{R}^d) \iff V_g f \in \mathcal{S}^{(1)}(\mathbb{R}^d).
\]
For the proof we refer to [23, Theorems 3.8-9].

2.2. Modulation Spaces

The modulation space norms traditionally measure the joint time-frequency distribution of \( f \in \mathcal{S}' \), we refer, for instance, to [9], [12, Ch. 11-13] and the original literature quoted there for various properties and applications. In that setting it is sufficient to observe modulation spaces with weights which admit at most polynomial growth at infinity. However the study of ultra-distributions requires a more general approach that includes the weights of exponential growth.

Weight Functions. In the sequel \( v \) will always be a continuous, positive, even, submultiplicative function (submultiplicative weight), i.e., \( v(0) = 1, v(z) = v(-z) \), and \( v(z_1 + z_2) \leq v(z_1)v(z_2) \), for all \( z, z_1, z_2 \in \mathbb{R}^d \). Moreover, \( v \) is assumed to be even in each group of coordinates, that is, \( v(x, \omega) = v(-\omega, x) = v(-x, \omega) \), for any \( (x, \omega) \in \mathbb{R}^d \). Submultiplicativity implies that \( v(z) \) is dominated by an exponential function, i.e.
\[
(2.5) \quad \exists C, k > 0 \text{ such that } v(z) \leq C e^{k\|z\|}, \quad z \in \mathbb{R}^d.
\]

For example, every weight of the form \( v(z) = e^{a\|z\|^b}(1 + \|z\|^a) \log^r (e + \|z\|) \) for parameters \( a, r, s \geq 0, 0 \leq b \leq 1 \) satisfies the above conditions.

Associated to every submultiplicative weight we consider the class of so-called \( v \)-moderate weights \( \mathcal{M}_v \). A positive, even weight function \( m \) on \( \mathbb{R}^d \) belongs to \( \mathcal{M}_v \) if it satisfies the condition
\[
m(z_1 + z_2) \leq Cv(z_1)m(z_2) \quad \forall z_1, z_2 \in \mathbb{R}^d.
\]

We note that this definition implies that \( \frac{1}{v} \lesssim m \lesssim v, \ m \neq 0 \) everywhere, and that \( 1/m \in \mathcal{M}_v \).

For our investigation of localization operators we will mostly use the exponential weights defined by
\[
(2.6) \quad w_s(z) = w_s(x, \omega) = e^{s\|z\|}, \quad z = (x, \omega) \in \mathbb{R}^d,
\]
\[
(2.7) \quad \tau_s(z) = \tau_s(x, \omega) = e^{s\|\omega\|}.
\]

Notice that arguing on \( \mathbb{R}^d \) we may read
\[
(2.8) \quad \tau_s(z, \zeta) = w_s(\zeta) \quad z, \zeta \in \mathbb{R}^d,
\]
which will be used in the sequel.

**Definition 2.1.** Given a non-zero window \( g \in \mathcal{S}^{(1)}(\mathbb{R}^d) \), a weight function \( m \in \mathcal{M}_v \) defined on \( \mathbb{R}^d \), and \( 1 \leq p, q \leq \infty \), the modulation space \( M^{p,q}_{m} (\mathbb{R}^d) \) consists of all tempered ultra-distributions \( f \in \mathcal{S}^{(1)}(\mathbb{R}^d) \) such that \( V_g f \in L^{p,q}_{m}(\mathbb{R}^d) \) (weighted mixed-norm spaces). The norm on \( M^{p,q}_{m} \) is

\[
\|f\|_{M^{p,q}_{m}} = \|V_g f\|_{L^{p,q}_{m}} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_g f(x, \omega)|^p m(x, \omega)^p \, dx \right)^{q/p} \, d\omega \right)^{1/q},
\]

(obvious changes if \( p = \infty \) or \( q = \infty \)).

Note that, for \( f, g \in \mathcal{S}^{(1)}(\mathbb{R}^d) \) the above integral is convergent thanks to (2.4). Namely, in view of (2.5) for a given \( m \) by (2.4), and therefore

\[
\int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_g f(x, \omega)|^p m(x, \omega)^p \, dx \right)^{q/p} \, d\omega \leq C \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_g f(x, \omega)|^p m(x, \omega)^p \, dx \right)^{q/p} \, d\omega \right)^{1/q} < \infty
\]

since by (2.4) and Theorem 2.2, d) we have \( |V_g f(x, \omega)| < C e^{-s\|x, \omega\|} \) for every \( s > 0 \). This implies \( \mathcal{S}^{(1)}(\mathbb{R}^d) \subset M^{p,q}_{m} \).

If \( p = q \), we write \( M^{p,p}_{m} \) instead of \( M^{p,q}_{m} \), and if \( m(z) \equiv 1 \) on \( \mathbb{R}^d \), then we write \( M^{p,q} \) and \( M^{p,p} \) for \( M^{p,q}_{m} \) and \( M^{p,q}_{m} \), respectively.

In the next proposition we show that \( M^{p,q}_{m} (\mathbb{R}^d) \) are Banach spaces whose definition is independent of the choice of the window \( g \in M^{1}_{v} \setminus \{0\} \). In order to do so, we need the adjoint of the short-time Fourier transform.

For given window \( g \in \mathcal{S}^{(1)} \) and a function \( F(x, \xi) \in L^{p,q}_{m}(\mathbb{R}^{2d}) \) we define \( V_g^* F \) by

\[
\langle V_g^* F, f \rangle := \langle F, V_g f \rangle,
\]

whenever the duality is well defined.

In our context, [12, Proposition 11.3.2.] can be rewritten as follows.

**Proposition 2.2.** Fix \( m \in \mathcal{M}_v \) and \( g, \psi \in \mathcal{S}^{(1)} \), with \( \langle g, \psi \rangle \neq 0 \). Then

a) \( V_g^* : L^{p,q}_{m}(\mathbb{R}^{2d}) \to M^{p,q}_{m}(\mathbb{R}^d) \), and

\[
\|V_g^* F\|_{M^{p,q}_{m}} \leq C \|V_g \psi\|_{L^{p,q}_{m}^*} \|F\|_{L^{p,q}_{m}}.
\]

b) The inversion formula holds: \( I_{M^{p,q}_{m}} = \langle g, \psi \rangle^{-1} V_g^* V_g \), where \( I_{M^{p,q}_{m}} \) stands for the identity operator.

c) The definition of \( M^{p,q}_{m} \) is independent on the choice of \( g \in \mathcal{S}^{(1)} \setminus \{0\} \) and different windows yield equivalent norms.

d) The space of admissible windows can be extended from \( \mathcal{S}^{(1)} \) to \( M^{1}_{v} \).
Proof. a) Clearly,
\[ |\langle V^*_g F, f \rangle| = |\langle F, V_g f \rangle| \leq \|F\|_{L_{p,q}^s} \|V_g f\|_{L_{p',q'}^s}, \]
where \( p' = \frac{p}{p-1} \) and \( q' = \frac{2(1-q)}{2(1-q)-1} \). Moreover, there exists \( C > 0 \) such that \( 1/m \leq C/\nu \) which together with (2.5) implies that there exist \( l > 0 \) such that \( \|e^{-t|z|}\|_{L_{p',q'}^{s,l/m}} < \infty \). Now, \( f \in S^{(1)} \) implies that \( V_g f \in S^{(1)}(\mathbb{R}^d) \), and we obtain
\[ \|V^*_g F, f\| \leq \|F\|_{L_{p,q}^s} \|e^{l|z|}\|_{\infty} \|e^{-l|z|}\|_{L_{p',q'}^{s,l/m}} < \infty. \]

Therefore, \( V^*_g F \) is a well defined (tempered) ultra-distribution. Proceeding then as in the proof of [12, Proposition 11.3.2 a)] we come to the desired result.

b) It follows by the same arguments of [12, Proposition 11.3.2 b)] by replacing \( S' \) with \( S^{(1)} \).

c) Now that a) and b) are settled c) follows almost directly from the corresponding result [12, Proposition 11.3.2 c)].

d) For this part we need the density of \( S^{(1)} \) in \( M_{p,q}^n \). This fact is not obvious and will be proven in a more general setting in [4]. Then d) follows by using standard arguments of [12, Theorem 11.3.7].

REMARK: a) The preceding proofs motivate our initial choice of \( S^{(1)} \) for the window functions, instead of \( S^{(1)} \). More precisely, in the case of the inductive limit of Gelfand-Shilov type spaces, the short-time Fourier transform decays faster than \( e^{l_0|z|^s} \) for certain \( l_0 > 0 \). This would not be sufficient to ensure \( \|e^{l|z|}\|_{\infty} < \infty \) in (2.10). However, the space \( S^{(1)} \) will play an important role in the last section, see Corollary 4.3.

b) Proposition 2.2 d) actually implies that Definition 2.1 coincides with the usual definition of modulation spaces with weights of polynomial and sub-exponential growth (see, for example [5, 9, 12, 19]). We do not prove that \( M_{p,q}^n \) are Banach spaces. This follows from Proposition 2.2 and arguments used in [12, Theorem 11.3.5].

To benefit of non-expert readers, we recall that the class of modulation spaces contains the following well-known function spaces:

Weighted \( L^2 \)-spaces: \( M^2_{(s)}(\mathbb{R}^d) = L^2_{(s)}(\mathbb{R}^d) = \{ f : f(x)x^s \in L^2(\mathbb{R}^d) \}, s \in \mathbb{R}. \)

Sobolev spaces: \( M^2_{(s)}(\mathbb{R}^d) = H^s(\mathbb{R}^d) = \{ f : f(\omega)\omega^s \in L^2(\mathbb{R}^d) \}, s \in \mathbb{R}. \)

Shubin-Sobolev spaces [22, 2]: \( M^2_{(s,\omega)}(\mathbb{R}^d) = L^2_{(s)}(\mathbb{R}^d) \cap H^s(\mathbb{R}^d) = Q_s(\mathbb{R}^d). \)

Feichtinger’s algebra: \( M^1(\mathbb{R}^d) = S_0(\mathbb{R}^d). \)

The characterization of the Schwartz class given in [14]: \( S(\mathbb{R}^d) = \bigcap_{s \geq 0} M^1_{(s)}(\mathbb{R}^d) \) and the corresponding description of the space of tempered distributions \( S'(\mathbb{R}^d) = \bigcup_{s \geq 0} M^1_{(s)}(\mathbb{R}^d) \) can now be reformulated for Gelfand-Shilov spaces and tempered ultra-distributions.
Proposition 2.3. Let \(1 \leq p, q \leq \infty\), and let \(w_s\) be given by (2.6). We have

\[ a) \quad S^{(1)} = \bigcap_{s \geq 0} M_{w_s}^{p,q}, \quad S^{(1)\prime} = \bigcup_{s \geq 0} M_{1/w_s}^{p,q}. \]

\[ b) \quad S^{(1)} = \bigcup_{s > 0} M_{w_s}^{p,q}, \quad S^{(1)\prime} = \bigcap_{s > 0} M_{1/w_s}^{p,q}. \]

The proof is just a rearrangement of [23, Theorem 4.3], written for the case of sub-exponential weights \(e^{s\|x,\omega\|_1^t}, s \geq 0\) and \(t > 1\). Here we observe the case \(t = 1\).

We finally recall a convolution relation for modulation spaces [6, Proposition 2.4].

Proposition 2.4. Let \(m \in M_v\) defined on \(\mathbb{R}^{2d}\) and let \(m_1(x) = m(x,0)\) and \(m_2(\omega) = m(0,\omega)\), the restrictions to \(\mathbb{R}^d \times \{0\}\) and \(\{0\} \times \mathbb{R}^d\), and likewise for \(v\). Let \(\nu(\omega) > 0\) be an arbitrary weight function on \(\mathbb{R}^d\) and \(1 \leq p, q, r, s, t \leq \infty\). If

\[ \frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}, \quad \text{and} \quad \frac{1}{t} + \frac{1}{r} = 1, \]

then

\[ M_{m_1 \otimes v}^{p,s,t}(\mathbb{R}^d) * M_{v_1 \otimes v_2 \nu^{-1}}^{q,s\prime,t\prime}(\mathbb{R}^d) \hookrightarrow M_m^{r,s}(\mathbb{R}^d) \]

with norm inequality \(\|f * h\|_{M_m^{r,s}} \lesssim \|f\|_{M_{m_1 \otimes v}^{p,s,t}} \|h\|_{M_{v_1 \otimes v_2 \nu^{-1}}^{q,s\prime,t\prime}}\).

3. Sufficient and necessary conditions

If we consider a ultra-distributional symbol \(a \in S^{(1)\prime}\), provided that the windows \(\varphi_1, \varphi_2\) belong to the smooth space \(S^{(1)}\), the localization operators \(A_{\varphi_1, \varphi_2} a\) is a well-defined and bounded mapping \(A_{\varphi_1, \varphi_2} a : S^{(1)} \rightarrow S^{(1)\prime}\). This is easily seen by the characterization (2.4), the weak definition of a localization operator (1.5) and the fact that \(S^{(1)}\) is closed under multiplication.

Next, we study the regularity properties of the operator \(A_{\varphi_1, \varphi_2} a\) on \(L^2\), within the framework of ultra-modulation spaces. This shall be achieved by passing through the connection between localization operators and Weyl operators. Namely, let us recall that the Weyl operator \(L_{\sigma}\), with symbol \(\sigma \in S^{(1)}(\mathbb{R}^{2d})\), is defined by

\[ \langle L_{\sigma} f, g \rangle = \langle \sigma, W(g,f) \rangle, \quad f, g \in S^{(1)}(\mathbb{R}^d), \]

where the cross-Wigner distribution \(W(f,g)\) is defined to be

\[ W(f,g)(x,\omega) = \int f(x + \frac{t}{2})g(x - \frac{t}{2})e^{-2\pi i\omega t} dt. \]

Every localization operator \(A_{\varphi_1, \varphi_2} a\) can be represented as a Weyl operator as follows [2, 7, 10, 22]: \(A_{\varphi_1, \varphi_2} a = L_{a * W(\varphi_2, \varphi_1)}\), that is the Weyl symbol of \(A_{\varphi_1, \varphi_2} a\) is given by (1.1).
We will use the following boundedness and Schatten class $S_p$ results for the Weyl calculus in terms of modulation spaces. For the proof we refer to [6, 13].

**Theorem 3.1.** a) If $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$, then $L_\sigma$ is bounded on $M^{p,q}(\mathbb{R}^d)$, $1 \leq p, q \leq \infty$, with a uniform estimate $\|L_\sigma\|_{op} \lesssim \|\sigma\|_{M^{\infty,1}}$ for the operator norm. In particular, $L_\sigma$ is bounded on $L^2(\mathbb{R}^d)$.

b) If $\sigma \in M^1(\mathbb{R}^{2d})$, then $L_\sigma \in S_1$ and $\|L_\sigma\|_{S_1} \lesssim \|\sigma\|_{M^1}$.

c) If $1 \leq p \leq 2$ and $\sigma \in M^p(\mathbb{R}^{2d})$, then $L_\sigma \in S_p$ and $\|L_\sigma\|_{S_p} \lesssim \|\sigma\|_{M^p}$.

d) If $2 \leq p \leq \infty$ and $\sigma \in M^{p,p'}(\mathbb{R}^{2d})$, then $L_\sigma \in S_p$ and $\|L_\sigma\|_{S_p} \lesssim \|\sigma\|_{M^{p,p'}}$.

We present a reformulation of [6, Proposition 2.5] in terms of modulation spaces. For the proof we refer to [6, 13].

**Lemma 3.1.** Let $w_s, \tau_s$ be the weights defined in (2.6) and (2.7). If $1 \leq p \leq \infty$, $s \geq 0$, $\varphi_1 \in M^p_{w_s}(\mathbb{R}^d)$ and $\varphi_2 \in M^p_{w_s}(\mathbb{R}^d)$, then $W(\varphi_2, \varphi_1) \in M^p_{\tau_s}(\mathbb{R}^{2d})$, and
\[
(3.3) \quad \|W(\varphi_2, \varphi_1)\|_{M^p_{\tau_s}} \lesssim \|\varphi_1\|_{M^p_{w_s}} \|\varphi_2\|_{M^p_{w_s}}.
\]

**Theorem 3.2.** a) Let $s \geq 0$, $a \in M^{\infty,1}_{\tau_s}(\mathbb{R}^{2d})$, $\varphi_1, \varphi_2 \in M^1_{w_s}(\mathbb{R}^d)$. Then $A^{a,\varphi_1,\varphi_2}$ is bounded on $M^{p,q}(\mathbb{R}^d)$ for all $1 \leq p, q \leq \infty$, and the operator norm satisfies the uniform estimate
\[
\|A^{a,\varphi_1,\varphi_2}\|_{op} \lesssim \|a\|_{M^{\infty,1}_{\tau_s}} \|\varphi_1\|_{M^1_{w_s}} \|\varphi_2\|_{M^1_{w_s}}.
\]

b) If $1 \leq p \leq 2$, then the mapping $(a, \varphi_1, \varphi_2) \mapsto A^{a,\varphi_1,\varphi_2}$ is bounded from $M^{p,\infty}_{\tau_s}(\mathbb{R}^{2d}) \times M^1_{w_s}(\mathbb{R}^d) \times M^p_{w_s}(\mathbb{R}^d)$ into $S_p$, in other words,
\[
\|A^{a,\varphi_1,\varphi_2}\|_{S_p} \lesssim \|a\|_{M^{p,\infty}_{\tau_s}} \|\varphi_1\|_{M^1_{w_s}} \|\varphi_2\|_{M^p_{w_s}}.
\]

c) If $2 \leq p \leq \infty$, then the mapping $(a, \varphi_1, \varphi_2) \mapsto A^{a,\varphi_1,\varphi_2}$ is bounded from $M^{p,\infty}_{\tau_s}(\mathbb{R}^d) \times M^1_{w_s} \times M^p_{w_s}$ into $S_p$, and
\[
\|A^{a,\varphi_1,\varphi_2}\|_{S_p} \lesssim \|a\|_{M^{p,\infty}_{\tau_s}} \|\varphi_1\|_{M^1_{w_s}} \|\varphi_2\|_{M^p_{w_s}}.
\]

**Proof.** a) We use the convolution relation (2.11) to show that the Weyl symbol $a \ast W(\varphi_2, \varphi_1)$ of $A^{a,\varphi_1,\varphi_2}$ is in $M^{\infty,1}$. If $\varphi_1, \varphi_2 \in M^1_{w_s}(\mathbb{R}^d)$, then by (3.3), we have $W(\varphi_2, \varphi_1) \in M^1_{\tau_s}(\mathbb{R}^{2d})$. Applying Proposition 2.4 in the form $M^1_{\tau_s} \ast M^1_{\tau_s} \subseteq M^{\infty,1}$, we obtain $\sigma = a \ast W(\varphi_2, \varphi_1) \in M^{\infty,1}$. The result now follows from Theorem 3.1 a).

Similarly, the proof of b) and c) is based on results of Proposition 2.4 and Theorem 3.1, items b) - d). \[\blacksquare\]

For the sake of completeness, we state the necessary boundedness result, which follows by straightforward modifications of [6, Theorem 4.3].
**Theorem 3.3.** Let \( a \in \mathcal{S}^{(1)'}(\mathbb{R}^d) \) and \( s \geq 0 \). If there exists a constant \( C = C(a) > 0 \) depending only on \( a \) such that
\[
\| A_a^{\varphi_1,\varphi_2} \|_{S_s^\infty} \leq C \| \varphi_1 \|_{M_1} \| \varphi_2 \|_{M_1}
\]
for all \( \varphi_1, \varphi_2 \in \mathcal{S}^{(1)}(\mathbb{R}^d) \), then \( a \in M_{1/s}^\infty \).

4. Ultra-distributions with Compact Support as Symbols

As an application we treat ultra-distributions with compact support, denoted here by \( \mathcal{E}'_t \), \( t > 1 \). The Gelfand-Shilov space \( \mathcal{S}') \) is embedded in the corresponding Gevrey function space and therefore in view of Lemma 2.1 we have \( \mathcal{E}'_t \subset \mathcal{S}^{(1)'} \subset \mathcal{S}'(1) \). Hence, an element from \( \mathcal{E}'_t \), \( t > 1 \) can be viewed as an element of \( \mathcal{S}^{(1)'} \). We skip the precise definition of \( \mathcal{E}'_t \), see e.g. [20, Definition 1.5.5, Example 1.8.4 (b)]. In fact, the following structure theorem, obtained by a slight generalization of [20, Theorem 1.5.6] (see also [21]) to the anisotropic case, will be sufficient for our purposes.

**Theorem 4.1.** Let \( t \in \mathbb{R}^d, t > 1 \). Every \( u \in \mathcal{E}'_t(\mathbb{R}^d) \) can be represented as
\[
u = \sum_{\alpha \in \mathbb{N}_0^d} \partial^\alpha \mu_\alpha,
\]
where \( \mu_\alpha \) is a measure satisfying, for every \( \epsilon = (\epsilon_1, \ldots, \epsilon_d) > 0 \),
\[
\int_K |d\mu_\alpha| \leq C_\epsilon \epsilon^\alpha(\alpha!)^{-t},
\]
for a suitable constant \( C_\epsilon > 0 \) and a suitable compact set \( K \subset \mathbb{R}^d \), independent of \( \alpha \).

**Proposition 4.2.** Let \( t \in \mathbb{R}^d, t > 1 \), and \( a \in \mathcal{E}'_t(\mathbb{R}^d) \). Then its STFT with respect to any window \( g \in \mathcal{S}^{(1)} \) satisfies the estimate
\[
|V_g a(x, \omega)| \lesssim e^{-h \cdot |x|} e^{t \cdot |2\pi \omega|^{1/t}},
\]
for every \( h > 0 \).

**Proof.** Note that, for every \( g \in \mathcal{S}^{(1)} \), we have
\[
\partial^\alpha (M_\omega T_s g)(y) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (2\pi i \omega)^\beta M_\omega T_s \partial_y^{\alpha-\beta} g(y).
\]
Then, using the representation (4.1), the STFT of $a \in \mathcal{E}'_t$ can be estimated as follows:

$$|V_g a(x, \omega)| \leq \sum_{\alpha \in \mathbb{N}_0^d} |\langle \partial^\alpha \mu_\alpha, M_\omega T_x g \rangle| = \sum_{\alpha \in \mathbb{N}_0^d} |\langle \mu_\alpha, \partial^\alpha (M_\omega T_x g) \rangle| \leq \sum_{\alpha \in \mathbb{N}_0^d} \sum_{\beta \leq \alpha} \left(\frac{\alpha}{\beta}!\right) \left|\left(2\pi i \omega\right)^{\beta} \int_{\mathbb{R}^d} (M_\omega T_x \partial^\alpha \partial^{-\beta} g)(y) |d\mu_\alpha(y)|.\right.$$  

Now we use the estimate (2.2), where we may assume $B_j = 1$ for all $j = 1, \ldots, d$, to get:

$$|M_\omega T_x \partial^\alpha \partial^{-\beta} g(y)| = |T_x \partial^\alpha \partial^{-\beta} g(y)| \leq C(\alpha - \beta)! e^{-h \cdot |y - x|},$$

for every $h > 0$. Furthermore, recall $\sum_{\beta \leq \alpha} \left(\frac{\alpha}{\beta}!\right) = 2^{|\alpha|_1}$, so that we have

$$|V_g a(x, \omega)| \lesssim \sum_{\alpha \in \mathbb{N}_0^d} 2^{|\alpha|_1} \sum_{\beta \leq \alpha} \left|\frac{2\pi \omega |\beta|}{(\beta!)^t} ((\alpha - \beta)!)^t \int_{\mathbb{R}^d} e^{-h \cdot |y - x|} |d\mu_\alpha(y)|.\right|$$

From the representation theorem (4.2) we get, for every $\epsilon > 0$:

$$\int e^{-h \cdot |y - x|} |d\mu_\alpha(y)| \leq e^{-h \cdot |x|} \int e^{h \cdot |y|} |d\mu_\alpha(y)| \leq C_{h,K} e^{-h \cdot |x|} e^{\alpha} (\alpha!)^{-t}.$$  

Since $\beta!(\alpha - \beta)! \leq \alpha!$, we obtain

$$|V_g a(x, \omega)| \lesssim e^{-h \cdot |x|} \sum_{\alpha \in \mathbb{N}_0^d} (2\epsilon)^{\alpha} \sum_{\beta \leq \alpha} \left|\left[\frac{2\pi \omega |\beta|}{\beta!}\right]^t\right|,$$

for every $\epsilon > 0$. We then choose $\epsilon < 1/2$ and the following estimate assures the desired result:

$$\sum_{\beta \leq \alpha} \left|\left[\frac{2\pi \omega |\beta|}{\beta!}\right]^t\right| \leq \sum_{\beta \in \mathbb{N}_0^d} \left|\left[\frac{2\pi \omega |\beta|}{\beta!}\right]^t\right| \leq \left(\sum_{\beta \in \mathbb{N}_0^d} \left|\frac{2\pi \omega |\beta|}{\beta!}\right|^t\right)^{\frac{1}{t}} = e^{t \cdot |2\pi \omega|^{1/t}}.$$  

\[ \blacksquare \]

**Corollary 4.3.** Let $t \in \mathbb{R}^d$, $t > 1$ and $a \in \mathcal{E}'(\mathbb{R}^{2d})$. If the window functions $\varphi_1, \varphi_2 \in S^{(1)}(\mathbb{R}^d)$, then $A_{\varphi_1, \varphi_2}$ is a trace class operator.

**Proof.** If $\varphi_1, \varphi_2 \in S^{(1)}(\mathbb{R}^d)$, Proposition 2.3 b), with $p = q = 1$, assures the existence of an $\epsilon > 0$ such that $\varphi_1, \varphi_2 \in M^1_{\omega}(\mathbb{R}^d)$. On the other hand, for $\|\omega\| > C_{\epsilon}$ (where $C_{\epsilon}$ is a suitable positive constant depending on $\epsilon$) we can write

$$t \cdot |2\pi \omega|^{1/t} = \sum_{i=1}^d t_i |2\pi \omega_i|^{1/t_i} \leq \epsilon \|\omega\|;$$
then, the estimate of Proposition 4.2 gives $a \in M_{1/\tau, \infty}^{1\infty}(\mathbb{R}^{2d})$. Finally, since $\varphi_1, \varphi_2 \in M_{1, \tau}^1(\mathbb{R}^d)$ and $a \in M_{1/\tau, \infty}^{1\infty}(\mathbb{R}^{2d})$, Theorem 3.2 b), written for the case $p = 1$, implies that the operator $A_{\varphi_1, \varphi_2}$ is a trace class operator.

References


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