1 CW-complexes

There are two slightly different (but of course equivalent) definitions of a CW-complex. Basically, one version is suitable when you have a given space and want to provide it with a CW-structure, the other one is better when you want to construct a space (with structure).

It’s fair to say that the concept of a CW-complex is what you need to do nice inductive arguments in algebraic topology.

**Definition 1.1** A CW-complex is a Hausdorff space $X$ with a partition into disjoint subset, called (open) cells, typically denoted $e_n$ where $n$ runs through some indexing set (usually not denoted by anything). The requirements on this partition are as follows:

1. For each cell $e_n$ there is a map (not part of the structure) $\Phi_n : D^n \to X$, such that $\Phi_n : D^n - S^{n-1} \to e_n$ is a homeomorphism.

   Here $\Phi_n$ is said to be a characteristic map for the cell $e_n$. The restriction $\phi_n$ of $\Phi_n$ to $S^{n-1}$ is an attaching map for the cell. The integer $n \geq 0$ is the dimension of the cell, $\dim e_n$, and is uniquely determined since $D^n - S^{n-1}$ is homeomorphic to $D^m - S^{m-1}$ only if $m = n$, being open subsets of Euclidean spaces. If $e_n$ is known to have dimension $n$ it’s usually denoted by $e_n^n$.

   Since $D^n$ is compact and $X$ Hausdorff $\Phi_n(D^n)$ is closed, so $\bar{e_n} \subset \Phi_n(D^n)$. On the other hand $\Phi_n^{-1}(\bar{e_n})$ is closed and contains $D^n - S^{n-1}$, hence $\Phi_n^{-1}(\bar{e_n}) = D^n$, showing that $\Phi_n(D^n) \subset \bar{e_n}$. The map $\Phi_n : D^n \to \bar{e_n}$ is a quotient map ($F \subset \bar{e_n}$ is closed iff $\Phi_n^{-1}(F)$ is closed) since $D^n$ is compact and $\bar{e_n}$ is Hausdorff.

2. For each cell $e_n$ the set $\bar{e_n} - e_n$ is contained in a finite union of cells of dimension $< n$.

   This property is called closure finiteness and accounts for the C in CW.

3. A subset $F$ of $X$ is closed iff $F \cap \bar{e_n}$ is closed for each cell $e_n$ of $X$.

   This is expressed by saying that $X$ has the weak topology with respect to the closures of the cells and explains the W in CW.

Note that if $X$ is a finite CW-complex (i.e there are only finitely many cells in the partition) property 2. reduces to: $\bar{e_n} - e_n$ is contained in a union of cells of dimension $< n$ and property 3. is automatically satisfied.

**Example 1.1** We give $S^n$ a CW-structure by partitioning it inductively. Assume $S^{n-1}$ is partitioned, then partition the rest of $S^n$ into the upper and lower (open) hemispheres. Characteristic maps for these two additional cells can be taken to be the obvious maps from $D^n$ to the closed hemispheres of $S^n$. In this CW-structure there are two cells of each dimension $\leq n$.

Defining $S^\infty = \bigcup_n S^n$ and giving it a topology by defining $F \subset S^\infty$ to be closed iff $F \cap S^n$ is closed (in $S^n$) for each $n \geq 0$ we have a CW-structure on $S^\infty$.

Another CW-structure on $S^n$ is given by the partition consisting of $\{e_1\}$ ($e_1$ being the first vector in the standard basis of $\mathbb{R}^{n+1}$) and $S^n - \{e_1\}$. As a characteristic map for the large cell we can take a quotient map $D^n \to S^n$ identifying $S^{n-1}$ to $e_1$. In this CW-structure there is one cell of dimension 0 and one in dimension $n$ (and no others).
A CW-complex is of dimension $n$ if all cells have dimension $\leq n$ and there is at least one cell of dimension $n$. It is of finite type if there are only finitely many cells of each dimension.

A sub complex $A$ of a CW-complex $X$ is a union of cells $e$ in $X$ such that $e \subset A \Rightarrow \bar{e} \subset A$. In the first CW-structure on $S^n$ above $S^{n-1}$ is a sub complex of $S^n$ and $S^n$ is a sub complex of $S^\infty$.

Any union and intersection of sub complexes is a sub complex.

1. Any cell $e$ of $X$ is contained in a finite sub complex $K(e)$. This is proved by induction on the dimension of the cell.

A 0-dimensional cell is just a point and a sub complex. Assuming the result for cells of dimension $< n$, let $e^n$ be a cell of dimension $n$. Then $\bar{e}^n - e^n$ is contained in a finite union of cells of dimension $< n$. Each of these is contained in a finite sub complex. The union of these is a finite sub complex containing $\bar{e}^n - e^n$. Adjoining the cell $e^n$ results in a finite sub complex containing $e^n$.

A finite sub complex $B$ of $X$ is obviously closed being a finite union of the sets $\bar{e}$ for $e \subset B$. In fact:

2. Any sub complex $A$ of $X$ is closed.

To see this, let $e$ be a cell of $X$ and $K(e)$ a finite sub complex of $X$ containing $e$. Then $\overline{e} \cap A = \bar{e} \cap K(e) \cap A$. Here $\bar{e}$ is closed and so is $K(e) \cap A$, being a finite sub complex.

3. A sub complex $A$ of $X$ is a CW-complex.

In the definition of a CW-complex 1. and 2. are immediate. Suppose $F \subset A$ has $\bar{e}_\alpha \cap F$ closed for each cell $e_\alpha$ in $A$ and let $e$ be a cell of $X$ contained in the finite sub complex $K(e)$. Then $F \cap \bar{e} = F \cap \bar{e} \cap K(e) \cap A$. Here $K(e) \cap A$ is a finite sub complex of $X$ and hence closed in $A$. By assumption on $F$ the set $F \cap K(e) \cap A$ is closed, hence so is $F \cap \bar{e} = F \cap e \cap K(e) \cap A$.

Any finite sub complex of $X$ is a compact subspace, since it is a finite union of compact sets $\bar{e}$. In fact:

4. Any compact subset $K$ of $X$ is contained in a finite sub complex.

To see this, choose a point $x_\alpha \in K \cap e_\alpha$ for each cell such that this intersection is non-empty. Let $S$ be the subspace of all points so chosen and $T \subset S$. Then for any cell $e$ we have $T \cap \bar{e} = T \cap K(e) \cap \bar{e}$, where $K(e)$ is a finite sub complex containing $e$. Then $T \cap K(e)$ is finite and hence closed. Thus any subset of $S$ is closed and $S$ a discrete closed subspace of $K$ and consequently a discrete compact space. It follows that $S$ is finite and $K$ is contained in a finite union of cells and hence in a finite sub complex.

5. A map $f : X \to Y$ is continuous iff $f| : \bar{e} \to Y$ is continuous. If $\Phi : D^n \to \bar{e}$ is a characteristic map for $e$ then $f|$ is continuous iff $f| \circ \Phi : D^n \to Y$ is continuous.

The first follows immediately from the fact that $f^{-1}(F) \cap \bar{e} = (f|)^{-1}(F)$ and that $X$ has the weak topology with respect to the closure of its cells. The second is a consequence of $\Phi$ being a quotient map.

For each $k \geq 0$ let $X^k$ be the union of all cells of $X$ of dimension $\leq k$. This is a sub complex of $X$. Note that a cell $e^k$ of $X^k$ is an open subset of $X^k$. Indeed, the union of all other cells of $X^k$ is a sub complex and hence the complement of $e^k$ is closed.

Choose a characteristic map $\Phi_\alpha : D^k \to e^k_\alpha$ for each cell of dimension $k$ in $X^k$. They amalgamate to a map $\Phi : \coprod_\alpha D^k_\alpha \to X^k$ which restricts to a homeomorphism $\Phi| : \coprod_\alpha (D^k_\alpha - S^{k-1}) \to X^k - X^{k-1}$.

Let $X^k \to X^k/X^{k-1}$ be the quotient map collapsing $X^{k-1}$ to a point. Then the composite

$$\tilde{q} : \coprod_\alpha D^k_\alpha \xrightarrow{\Phi} X^k \xrightarrow{\bar{q}} X^k/X^{k-1}$$

is a closed. Indeed, if $F \subset \coprod_\alpha D^k_\alpha$ is closed then

$$q^{-1}(\tilde{q}(F)) = \begin{cases} 
\Phi(F) & \text{if } F \cap (\coprod_\alpha S^{k-1}_\alpha) = \emptyset \\
\Phi(F) \cup X^{k-1} & \text{if } F \cap (\coprod_\alpha S^{k-1}_\alpha) \neq \emptyset.
\end{cases}$$
and intersecting this with closures of cells of $X^k$ and using that each $\Phi_{\alpha}$ is a quotient map one sees that both alternatives gives closed sets in $X^k$, so $\overline{q}(F)$ is closed since $q$ is a quotient map. In particular $\overline{q}$ is a quotient map collapsing $\prod_{\alpha} S_{\alpha}^{k-1}$ to a single point, i.e.

$$\prod_{\alpha} D_{\alpha}^k \cong \prod_{\alpha} S_{\alpha}^{k-1} \cong X^k$$

On the other hand choose a quotient map $D^k \to S^k$ collapsing $S^{k-1}$ to a point $p \in S^k$. Taking disjoint union of spaces we get a quotient map $\prod_{\alpha} D_{\alpha}^k \to \prod_{\alpha} S_{\alpha}^k$ and collapsing the subspace consisting of all the points $p_{\alpha} \in S_{\alpha}^{k-1}$ to a point a quotient map $\prod_{\alpha} D_{\alpha}^k \to \bigvee_{\alpha} S_{\alpha}^k$ collapsing $\prod_{\alpha} S_{\alpha}^{k-1}$ to a singel point. It follows that

$$\bigvee_{\alpha} S_{\alpha}^k \cong \prod_{\alpha} D_{\alpha}^k \cong \prod_{\alpha} S_{\alpha}^{k-1} \cong X^k$$

We record this as

6. $X^k/X^{k-1}$ is (homeomorphic to) a wedge of spheres of dimension $k$ in one-to-one correspondens with the cells of dimension $k$.

Let $X$ is a CW-complex and $L$ a locally compact (Hasusdorff) space (each point in $L$ has an open neighborhood with compact closure, e.g. $L$ compact). Then:

7. A subset $F \subset X \times L$ is closed iff $F \cap (\overline{e} \times L)$ is closed for each cell $e$ of $X$.

Only $\implies$ is non trivial. So suppose $F \cap (\overline{e} \times L)$ is closed for each cell $e$ of $X$. Then $O = X \times L - F$ has $O \cap (\overline{e} \times L)$ open in $\overline{e} \times L$ for each cell $e$. Fix $(x_0, y_0) \in O$ and suppose $x_0 \in e_0$. Then $(x_0, y_0) \in O \cap (\overline{e}_0 \times L)$ and there is a neighborhood of $y_0$ in $L$ with compact closure $K$ such that $(x_0) \times K \subset O \cap (\overline{e}_0 \times L)$.

Let $A = \{x \in X \mid \{x\} \times K \subset O\}$. If we can prove that $A$ is open in $X$, then $A \times K$ is a neighborhood of $(x_0, y_0)$ in $X \times L$ and $O$ will be open and $F$ closed.

To check this let $e$ be a cell of $X$ and consider a point $x_1 \in A \cap \overline{e}$. Then $\{x_1\} \times K \subset O \cap (\overline{e} \times L)$ which is an open set of $\overline{e} \times L$. Compactness of $K$ gives an open set $O_{x_1}$ of $\overline{e}$ containing $x_1$ such that $O_{x_1} \times K \subset \overline{e} \times L$, i.e. $O_{x_1} \subset A \cap \overline{e}$ and $A \cap \overline{e}$ is open in $\overline{e}$. Since $X$ has the weak topology it follows that $A$ is open in $X$.

As a corollary of this we note:

8. A map $F : X \times L \to Y$ is continuous iff $F| : \overline{e} \times L \to Y$ is continuous for all cells $e$ of $X$.

Suppose now that $X$ and $Y$ are CW-complexes with cells $e_{\alpha}$ and $e_{\beta}$ respectively. Then $X \times Y$ is partitioned into sets $e_{\alpha} \times e_{\beta}$. Given characteristic maps $\Phi_{\alpha} : D^{n} \to X$ and $\Phi_{\beta} : D^{m} \to Y$ for these cells and choosing a homeomorhism $h : D^{n+m} \to D^{n} \times D^{m}$ we get a characteristic map $(\Phi_{\alpha} \times \Phi_{\beta}) \circ h$ for $e_{\alpha} \times e_{\beta}$.

The conditions 1. and 2. in the definition of a CW-complex is satisfied by this partition of $X \times Y$, but generally not the third one. The problem is that $X \times Y$ doesn’t necessarily have the weak topology with respect to the closures of the cells. One way to fix this is to simply givit the weak topology, but then this doesn’t agree with the product topology. The new space, which then is a CW-complex, is denoted $X \times_{c} Y$ and generally has more open set than $X \times Y$ has.

If $Y$ is locally compact it follows from item 7 that $X \times_{c} Y = X \times Y$. In particular this applies when $Y$ is a finite CW-complex.

**Example 1.2** If we give $S^5$ and $S^7$ the CW-structure with one cell in dimension 0 and one in dimensions 5 and 7 respectively, we have a CW-structure on $S^5 \times S^7$ with in total four cells of dimensions 0, 5, 7 and $5 + 7 = 12$. 

3