We undertake a more detailed study of minimal surfaces in this chapter. It builds on the theory of complex functions of one variable, the basics of which we take for granted. The concept of minimal isothermal patch, previously introduced in Section 16.7, allows us to start the present chapter by associating to a conjugate harmonic pair of patches an isometric deformation of minimal surfaces. The terminology of complex derivatives in Section 22.2 leads to the central notion of minimal curve in Section 22.3, and a simple algebraic method for constructing an isometric deformation from a minimal isothermal patch in Section 22.4.

The above methods are effective if one has a minimal surface to start with, but do not help otherwise. In 1866, Weierstrass\(^1\) achieved a major breakthrough in this regard. He gave a formula using complex variables that allows the easy generation of new minimal surfaces, and permits their investigation in a general manner. Weierstrass’s formula for generating minimal surfaces is described in Section 22.5. It is applied to some familiar examples, and also to generate a new class of minimal surfaces with one planar end.

Section 22.6 addresses the problem, first introduced on page 510, of constructing a minimal surface that incorporates a given plane curve as geodesic. This

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\(^{1}\)Karl Theodor Wilhelm Weierstrass (1815–1897). German mathematician. Weierstrass began his mathematical career as a high school teacher, but his work became recognized after he published a major paper on Abelian functions. This led to an appointment at the University of Berlin. His lectures in mathematics attracted students from all over the world. Topics of his lectures included mathematical physics, elliptic functions, and applications to problems in geometry and mechanics. Weierstrass’s introduction of the Weierstrass \(\wp\)-function revolutionized the theory of elliptic functions.
is achieved by means of a simple formula that first associates to a plane curve a minimal curve, that is then used to construct the required minimal surface. Examples of new minimal surfaces formed in this way include a generalization of the catenoid that has an ellipse instead of a circle as its center curve. The whole process can be effectively automated, as we shall see in Notebook 22.

Few complete minimal surfaces are embedded; for example, Enneper’s minimal surface has self-intersections in spite of its simple definition. Moreover, Weierstrass’s formula usually produces self-intersecting minimal surfaces. The four complete minimal surfaces without self-intersections that we have seen are: the catenoid, the helicoid, Scherk’s minimal surface and Scherk’s fifth minimal surface. Another much more complicated complete minimal surface with no self-intersections is Costa’s minimal surface, the combined beauty and simplicity of which has been the subject of attention from the artistic world. It is defined using the Weierstrass $\wp$ and $\zeta$ functions for a square lattice, rapidly introduced in the chapter’s final section. The plotting programs from Notebook 22 provide three images of Costa’s surface.

### 22.1 Isometric Deformations of Minimal Surfaces

Minimal surfaces and complex analytic functions have many properties in common. We quickly recall some pertinent facts regarding complex analytic functions of one variable. For more details see, for example, [Ahlf] or [Hille].

Let $\mathcal{U}$ be an open subset of $\mathbb{C}$. A function $f : \mathcal{U} \to \mathbb{C}$ is called complex analytic or holomorphic if $f$ has a derivative in the complex sense at each point $p$ of $\mathcal{U}$. This means that the limit
\[
\lim_{q \to p} \frac{f(q) - f(p)}{q - p}
\]
exists no matter how $q$ approaches the fixed point $p$ in the complex plane. In this case, if we write $g = \Re f$ and $h = \Im f$, then $g$ and $h$ are related by the Cauchy–Riemann equations
\[
22.1 \quad g_u = h_v, \quad g_v = -h_u.
\]
These are deduced by first letting $q$ tend to $p$ parallel to the real $x$-axis, and comparing the result with that when $q$ tends to $p$ parallel to the imaginary $y$-axis. Basic theorems from introductory complex analysis assert that a function

---

2 Augustin Louis Cauchy (1789–1857). One of the leading French mathematicians of the first half of the 19th century. He introduced rigor into calculus, including precise definitions of continuity and integration. His best known results include the Cauchy integral formula and Cauchy–Riemann equations in complex analysis, and the Cauchy-Kovalevskaya existence theorem for the solution of partial differential equations. Cauchy’s scientific output was enormous, second only to that of Euler.
22.1. ISOMETRIC DEFORMATIONS

Let \( f \) satisfying (22.1) on an open set \( U \subseteq \mathbb{C} \) not only has derivatives of all orders on \( U \), but has a power series expansion on any disk in \( U \).

Using the symmetry of the mixed second partial derivatives, we see that (22.1) implies that

\[
g_{uu} + g_{vv} = h_{uu} + h_{vv} = 0,
\]

so that that \( g \) and \( h \) are harmonic. For this reason, one can say that a pair \( g, h \) of real-valued functions of two variables are conjugate harmonic when (22.1) holds. These notions extend in a coordinate-wise way to \( n \)-tuples of functions:

**Definition 22.1.** Let \( U \) be an open subset of \( \mathbb{R}^2 \), and let \( x, y : U \to \mathbb{R}^n \) be patches. We say that

(i) \( x \) is harmonic if \( x_{uu} + x_{vv} = 0 \).

(ii) \( x \) and \( y \) are conjugate harmonic if

(22.2)

\[
x_u = y_v \quad \text{and} \quad x_v = -y_u.
\]

It again follows that (ii) implies (i). We shall discuss the problem of determining the conjugate \( y \) of a given harmonic patch \( x \) in Section 22.4.

As an immediate consequence of Lemma 16.18, page 520, we obtain

**Corollary 22.2.** A regular isothermal patch \( x : U \to \mathbb{R}^3 \) is a minimal surface if and only if it is harmonic.

So far we have only defined the notion of minimal surface in \( \mathbb{R}^3 \). Corollary 22.2 provides one way of extending this definition so as to talk of a minimal surface in \( \mathbb{R}^n \) for any \( n \) (another way is given in Exercise 11):

**Definition 22.3.** A minimal isothermal patch \( x : U \to \mathbb{R}^n \) is a patch that is both isothermal and harmonic.

Next, we describe a general method for obtaining a 1-parameter family of isometric minimal patches. This construction generalizes the deformation between the helicoid and the catenoid, and (16.8) on page 505 will assist in understanding the general case below:

**Definition 22.4.** Let \( x, y : U \to \mathbb{R}^n \) be conjugate harmonic minimal isothermal patches. The associated family of \( x \) and \( y \) is the 1-parameter family of patches \( z[t] : U \to \mathbb{R}^n \) given by

(22.3)

\[
z[t] = \Re \left( e^{-it}(x + iy) \right) = x \cos t + y \sin t.
\]

The last expression is of course equal to \((\cos t)x + (\sin t)y\), but writing scalars on the right will save us from inserting lots of parentheses below. First, we compute the derivatives of \( z[t] \).
Lemma 22.5. Let \( \mathbf{x}, \mathbf{y}: \mathcal{U} \to \mathbb{R}^n \) be conjugate harmonic minimal isothermal patches. Then the associated family \( t \mapsto \mathbf{z}[t] \) of \( \mathbf{x} \) and \( \mathbf{y} \) satisfies

\[
\begin{align*}
\mathbf{z}[t]_u &= \mathbf{x}_u \cos t - \mathbf{x}_v \sin t, \\
\mathbf{z}[t]_v &= \mathbf{x}_u \sin t + \mathbf{x}_v \cos t, \\
\mathbf{z}[t]_{uu} &= \mathbf{x}_{uu} \cos t - \mathbf{x}_{uv} \sin t = -\mathbf{z}[t]_{uv}, \\
\mathbf{z}[t]_{uv} &= \mathbf{x}_{uu} \sin t + \mathbf{x}_{uv} \cos t.
\end{align*}
\]

Furthermore, \( \mathbf{z}[t] \) and \( \mathbf{z}[t + \pi/2] \) are conjugate harmonic for any \( t \).

Proof. Equations (22.4) are a consequence of the Cauchy–Riemann equations and the assumption that \( \mathbf{x} \) and \( \mathbf{y} \) are isothermal. The last statement of the lemma can be proved as follows:

\[
\begin{align*}
\mathbf{z}[t]_u &= \Re (e^{-it}(\mathbf{x}_u + i\mathbf{y}_u)) = \Re (e^{-it}(\mathbf{y}_v - i\mathbf{x}_u)) \\
&= \Re (e^{-it(t+\pi/2)}(\mathbf{x}_v + i\mathbf{y}_v)) = \mathbf{z}[t + \pi/2]_v.
\end{align*}
\]

Similarly, \( \mathbf{z}[t]_v = -\mathbf{z}[t + \pi/2]_u \).

Figure 22.1 illustrates the patch (22.3), in which \( \mathbf{x} \) is Catalan’s surface shown in Figure 16.4 on page 510, and \( t = 3\pi/8 \). The snag is that the techniques of this section are not yet sufficient to find the equation of this deformation.
22.1. ISOMETRIC DEFORMATIONS

make the plot, it is necessary to find the conjugate patch \( y \), and the resulting complexification \( x + iy \) of \( x \); this was done automatically in Notebook 22 using the recipe given in Section 22.4 below.

Lemma 22.5 is a useful tool for obtaining information about the first and second fundamental forms of an associated family.

**Theorem 22.6.** Let \( x, y : U \to \mathbb{R}^n \) be conjugate harmonic minimal isothermal patches, and let \( t \mapsto z[t] \) be the associated family defined by (22.3). Then \( z[t] \) is a minimal isothermal patch for each \( t \), and all of the patches in the family have the same first fundamental form.

We say that \( z[t] \) is an isometric deformation from \( x \) to \( y \).

**Proof.** That \( z[t] \) is harmonic is a consequence of the third equation of (22.4). Let \( E(t), F(t), G(t) \) denote the coefficients of the first fundamental form of \( z[t] \). We use the first two equations of (22.4). Firstly,

\[
E(t) = z[t]_u \cdot z[t]_v = (x_u \cos t - x_v \sin t) \cdot (x_u \cos t - x_v \sin t).
\]

Since \( x_u \cdot x_u = x_v \cdot x_v = E \) and \( x_u \cdot x_v = 0 \), we obtain \( E(t) = x_u \cdot x_u = E \).

Similarly, \( F(t) = 0 \) and \( G(t) = x_v \cdot x_v = G \) for all \( t \). It follows that each \( z[t] \) is isothermal, and has the same first fundamental form as \( x \).

Next, we observe that, up to parallel translation, all the members of an associated family have the same unit normal and tangent space.

**Lemma 22.7.** Let \( x, y : U \to \mathbb{R}^3 \) be conjugate harmonic minimal isothermal patches.

(i) The unit normal \( U(t) \) of the patch \( z[t] \) at \( z[t](u,v) \) is parallel to the unit normal \( U \) of the patch \( x \) at \( x(u,v) \).

(ii) The tangent space to the patch \( z[t] \) at \( z[t](u,v) \) is parallel to the tangent space to the patch \( x \) at \( x(u,v) \).

**Proof.** Part (i) follows from the formula

\[
z[t]_u \times z[t]_v = x_u \times x_v, \quad t \in \mathbb{R},
\]

itself an easy consequence of (22.4). Obviously, (i) implies (ii).

In the light of (ii), we can identify the unit normal of \( z[t] \) with the unit normal \( U \) of \( x \). Thus

**Corollary 22.8.** All members of an associated family have the same Gauss map.

In spite of the above results, the shape operator of \( z[t] \) will in general depend on \( t \). Let \( e(t), f(t), g(t) \) denote the coefficients of the second fundamental form of \( z[t] \), and let \( S(t) \) denote the corresponding shape operator.
Lemma 22.9. The second fundamental form of \( z[t] : U \to \mathbb{R}^3 \) is related to that of \( x \) by the formulas

\[
\begin{align*}
    e(t) &= -g(t) = e \cos t - f \sin t, \\
    f(t) &= e \sin t + f \cos t.
\end{align*}
\]  

(22.5)

Furthermore,

\[
\begin{align*}
    S(t)z[t]_u &= Sx_u, \\
    S(t)z[t]_v &= Sx_v.
\end{align*}
\]  

(22.6)

Proof. We use the third equation of (22.4) to compute

\[
e(t) = z[t]_{uu} \cdot U(t) = (x_{uu} \cos t - x_{uv} \sin t) \cdot U = e \cos t - f \sin t,
\]

and the rest of (22.5) also follows from (22.4).

Combining the Weingarten equations (16.32), page 522, for the isothermal patch \( z[t] \) and Theorem 22.6 with (22.4) and (22.5), we get

\[
\begin{align*}
    \lambda^2 S(t)z[t]_u &= e(t)z[t]_u + f(t)z[t]_v \\
    &= (e \cos t - f \sin t)(x_u \cos t - x_v \sin t) \\
    &\quad + (e \sin t + f \cos t)(x_u \sin t + x_v \cos t) \\
    &= e x_u + f x_v = Sx_u.
\end{align*}
\]  

(22.7)

Similarly, \( S(t)z[t]_v = Sx_v \). □

We are now in a position to state a simple formula for the shape operator \( S(t) \) in terms of shape operator \( S \) and complex structure \( J \) of the minimal isothermal patch \( x \).

Lemma 22.10. The shape operator \( S(t) \) of \( z[t] \) is related to the shape operator \( S \) of \( x \) by the formula

\[
S(t) = (\cos t)S + (\sin t)S \circ J.
\]  

(22.8)

In particular, the shape operator \( S(\frac{\pi}{2}) \) of the conjugate patch of \( x \) equals the composition \( S \circ J \).

Proof. From (22.7) and (22.4), we get

\[
\begin{align*}
    Sx_u &= S(t)x_u \cos t - S(t)x_v \sin t, \\
    Sx_v &= S(t)x_u \sin t + S(t)x_v \cos t.
\end{align*}
\]

Solving for \( S(t)x_u \) and \( S(t)x_v \), and setting \( x_v = Jx_u \), we obtain

\[
\begin{align*}
    S(t)x_u &= Sx_u \cos t + SJx_u \sin t, \\
    S(t)x_v &= SJx_u \sin t + Sx_v \cos t.
\end{align*}
\]  

(22.9)

Clearly, (22.9) is equivalent to (22.8). □
Corollary 22.11. Let $x, y: \mathcal{U} \to \mathbb{R}^3$ be conjugate harmonic minimal isothermal patches, and let $v_p$ be a tangent vector to $x$ (which by Lemma 22.7 can also be considered to be a tangent vector to $y$). Then

(i) $v_p$ is an asymptotic vector of $x$ if and only if it is a principal vector of $y$;

(ii) $v_p$ is a principal vector of $x$ if and only if it is an asymptotic vector of $y$.

For proofs, see Exercise 1.

22.2 Complex Derivatives

Corollary 22.2 is the reason why harmonic functions are useful for the study of minimal surfaces. Since analytic function theory is so important for the study of harmonic functions of two variables, it is natural to introduce this subject into the study of minimal surfaces in $\mathbb{R}^n$.

It is often advantageous to switch from the standard coordinates $u, v$ in $\mathbb{R}^2$ that we have been using to complex coordinates $z$ and $\overline{z}$. The algebraic formulas relating the two kinds of coordinates are simple:

\[
\begin{align*}
\{ z = u + iv, \\
\overline{z} = u - iv,
\end{align*}
\]

\[
\begin{align*}
u = \frac{z + \overline{z}}{2}, \\
v = \frac{z - \overline{z}}{2i}.
\end{align*}
\]

(22.10)

(22.11)

It is a little disconcerting at first to work with $z$ and $\overline{z}$, because one tends to think that $z$ determines $\overline{z}$. This is certainly the case when we start with $u$ and $v$ and define $z$ and $\overline{z}$ by (22.10). But we can regard $z$ and $\overline{z}$ as abstract coordinates for $\mathbb{R}^2 = \mathbb{C}$ (even though they are complex-valued), and then define $u$ and $v$ by (22.11). This point of view frequently leads to simpler formulas. Thus, we shall see that $\{z, \overline{z}\}$ forms a coordinate system for $\mathbb{R}^2$ that we can use in place of the standard coordinate system $\{u, v\}$.

It is convenient to introduce the differential operators

\[
\begin{align*}
\frac{\partial}{\partial z} &= \tfrac{1}{2} \left( \frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}} = \tfrac{1}{2} \left( \frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right).
\end{align*}
\]

(22.12)

The motivation is that these operators satisfy the equations

\[
\begin{align*}
\frac{\partial}{\partial z}(z) &= 1, \quad \frac{\partial}{\partial z}(\overline{z}) = 0 \\
\frac{\partial}{\partial \overline{z}}(z) &= 0, \quad \frac{\partial}{\partial \overline{z}}(\overline{z}) = 1.
\end{align*}
\]
The Cauchy–Riemann equations (22.1) can now be expressed in the form
\[ \frac{\partial}{\partial z} f(z, \overline{z}) = 0. \]
From (22.10), we also have
\[ dz = du + idv, \quad d\overline{z} = du - idv. \]
(22.13)
and can write
\[ |dz|^2 = dzd\overline{z} = du^2 + dv^2 \quad \text{and} \quad \Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} = 4\frac{\partial^2}{\partial z\partial \overline{z}}. \]
Even though a patch \( x \) in \( \mathbb{R}^n \) consists of an \( n \)-tuple of real functions, it is very useful in minimal surface theory to apply \( \partial/\partial z \) to \( x \).

**Definition 22.12.** The complex derivative of a patch \( x: U \to \mathbb{R}^n \) is given by
\[ \frac{\partial x}{\partial z} = \frac{1}{2}(x_u - ix_v), \]
where \( z = u + iv. \)
We shall write
\[ \frac{\partial x}{\partial z} = (\phi_1[x], \ldots, \phi_n[x]) = \frac{1}{2}\left( \frac{\partial x_1}{\partial u} - i\frac{\partial x_1}{\partial v}, \ldots, \frac{\partial x_n}{\partial u} - i\frac{\partial x_n}{\partial v} \right). \]
With this notation, it is an easy matter to check the identities
\[ 4\sum_{k=1}^{n} \phi_k[x]^2 = x_u \cdot x_u - x_v \cdot x_v - 2ix_u \cdot x_v = E - G - 2iF, \]
(22.15)
and
\[ 4\sum_{k=1}^{n} |\phi_k[x]|^2 = x_u \cdot x_u + x_v \cdot x_v = E + G. \]
(22.16)
The next theorem shows that an isothermal minimal patch gives rise to an \( n \)-tuple of complex analytic functions such that the sum of the squares of the components equals zero. As we shall see below, this alternative description of an isothermal patch yields much useful information, because it allows the use of powerful theorems from complex analysis.

**Theorem 22.13.** Let \( x: U \to \mathbb{R}^n \) be a patch. Then
(i) \( x \) is harmonic if and only if (22.14) is complex analytic;
(ii) \( x \) is isothermal if and only if \( \sum_{k=1}^{n} \phi_k[x]^2 = 0; \)
(iii) if $\mathbf{x}$ is isothermal, then $\mathbf{x}$ is regular if and only if $\sum_{k=1}^{n} |\phi_k[\mathbf{x}]|^2 \neq 0$.

Conversely, let $\mathcal{U}$ be simply connected and let $\phi_1, \ldots, \phi_n: \mathcal{U} \to \mathbb{C}^n$ be complex analytic functions satisfying

\begin{equation}
\sum_{k=1}^{n} \phi_k^2 = 0 \quad \text{and} \quad \sum_{k=1}^{n} |\phi_k|^2 \neq 0.
\end{equation}

Then there exists a regular minimal isothermal patch $\mathbf{x}: \mathcal{U} \to \mathbb{R}^n$ satisfying (22.14).

**Proof.** Part (i) follows from the fact that the Cauchy–Riemann equations for $\partial \mathbf{x}/\partial z$ are $x_{uu} + x_{vv} = 0$ and (the identity) $x_{uv} - x_{vu} = 0$. Then (ii) is a consequence of (22.15), and (iii) follows from (22.16).

To prove the converse, suppose that complex analytic functions $\phi_j$ are given satisfying (22.17). Put

\[ \mathbf{x} = \left( \Re \int \phi_1(z) \, dz, \ldots, \Re \int \phi_n(z) \, dz \right); \]

the contour integrals are well defined and complex analytic, and their real parts harmonic. Moreover, (22.15) implies that $\mathbf{x}$ is isothermal, and (22.16) implies that $\mathbf{x}$ is regular. Thus $\mathbf{x}$ is a regular minimal isothermal patch. \[ \square \]

We know that a pair of conjugate harmonic functions determines a complex analytic function. Indeed,

**Lemma 22.14.** Let $\mathbf{x}, \mathbf{y}: \mathcal{U} \to \mathbb{R}^n$ be conjugate harmonic minimal isothermal patches. Then the map $\mathbf{x} + i \mathbf{y}: \mathcal{U} \to \mathbb{C}^n$ is complex analytic, and

\[ 2 \frac{\partial \mathbf{x}}{\partial z} = \frac{\partial}{\partial z} (\mathbf{x} + i \mathbf{y}). \]

**Proof.** The first statement follows from the fact that $\mathbf{x} + i \mathbf{y}$ is an $n$-tuple of complex functions of one variable, each of which is complex analytic. For the second statement we compute

\begin{equation}
\frac{\partial}{\partial z} (\mathbf{x} + i \mathbf{y}) = \frac{1}{2} \left( \frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right) (\mathbf{x} + i \mathbf{y})
= \frac{1}{2} (x_u - i x_v + y_u + i y_v) = x_u - i x_v = 2 \frac{\partial \mathbf{x}}{\partial z}.
\end{equation}

This will be relevant in the next section, given the definition of the associated family on page 721.
22.3 Minimal Curves

In this section we look at functions that have some of the properties of the complex derivative of a minimal isothermal patch. First, we reformulate (22.17) as a definition, making use of the complex dot product that was defined on page 194.

Definition 22.15. Let $U$ be an open subset of $\mathbb{C}$. A minimal curve is a complex analytic function $\Psi : U \rightarrow \mathbb{C}^n$ such that

\[ \Psi'(z) \cdot \Psi'(z) = 0 \tag{22.19} \]

for $z \in U$. If in addition $\Psi'(z) \cdot \overline{\Psi'(z)}$ is never zero for $z \in U$, we say that $\Psi$ is a regular minimal curve.

Minimal curves were first studied by Lie\(^3\) in [Lie]. A minimal curve can be thought of as a generalization of a minimal isothermal patch. In a different sense, a minimal curve is a generalization of a real curve in $\mathbb{R}^n$, and a complex-valued parametrized curve satisfying (22.19) is often called isotropic.

It is clear from the preceding section that, given a minimal curve $\Psi : U \rightarrow \mathbb{C}^n$, the patches $x, y : U \rightarrow \mathbb{R}^n$ defined by

\[ x(u, v) = \Re \Psi(u + iv) \quad \text{and} \quad y(u, v) = \Im \Psi(u + iv) \tag{22.20} \]

are conjugate minimal isothermal patches. We therefore call the pair $x, y$ the conjugate minimal isothermal patches associated to $\Psi$. They also determine an associated family $z[t] : U \rightarrow \mathbb{R}^n$ by setting

\[ z[t](u, v) = \Re (e^{-it}\Psi(u + iv)), \tag{22.21} \]

exactly as in (22.3). We emphasize that $z[t](u, v)$ is a real object; it is not equal to $x + iy$ which is instead denoted by $\Psi$. Theorem 22.6 tells us that this is an isometric deformation, and that $z[t]$ is a minimal isothermal patch for each $t$.

Let us rewrite Lemma 22.5 in terms of minimal curves.

Corollary 22.16. Let $\Psi : U \rightarrow \mathbb{C}^n$ be a minimal curve. Then the associated family $t \mapsto z[t]$ of $\Psi$ satisfies

\[ z[t] + i z[t] = e^{-it} \Psi'. \tag{22.22} \]

Proof. This is where we use (22.18). $\square$

---

\(^3\) Marius Sophus Lie (1842–1899). Norwegian mathematician. Although Lie is best known for Lie groups and Lie algebras, he made many important contributions to differential equations, surface theory and the theory of contact transformations.
It will be convenient to write
\[
\frac{1}{2} \Psi' = \frac{\partial x}{\partial z} = \frac{1}{2} (x_u - ix_v) = (\phi_1, \ldots, \phi_n)
\]
for the complex derivative of a patch \( x: \mathcal{U} \rightarrow \mathbb{R}^n \). Then
\[
\frac{1}{2} \overline{\Psi'} = \frac{\partial x}{\partial \overline{z}} = \frac{1}{2} (x_u + ix_v) = (\overline{\phi_1}, \ldots, \overline{\phi_n}).
\]

Using this notation, we now study the Gauss map of a minimal curve in \( \mathbb{C}^3 \).

**Definition 22.17.** The Gauss map of a minimal curve \( \Psi \) is the Gauss map of any member of an associated family of \( \Psi \). As usual, we identify the Gauss map with the unit normal \( \mathbf{U} \).

The next lemma follows from (7.9) on page 195.

**Lemma 22.18.** Let \( \Psi: \mathcal{U} \rightarrow \mathbb{C}^3 \) be a minimal curve. Then
\[
-i \Psi' \times \overline{\Psi'} = 2 (\Im(\phi_2 \overline{\phi_3}), \Im(\phi_3 \overline{\phi_1}), \Im(\phi_1 \overline{\phi_2})) = \|\Psi'\|^2 \mathbf{U}.
\]

**Proof.** It follows from (7.8) that
\[
\|\Psi'\|^2 = \|\Psi' \times \overline{\Psi'}\| = 2 \|x_u \times x_v\|.
\]

Therefore,
\[
\mathbf{U} = \frac{x_u \times x_v}{\|x_u \times x_v\|} = \frac{(\Psi' \times \overline{\Psi'})/(2i)}{\|\Psi'\|^2/2} = -i \frac{\Psi' \times \overline{\Psi'}}{\|\Psi'\|^2} = \frac{2 (\Im(\phi_2 \overline{\phi_3}), \Im(\phi_3 \overline{\phi_1}), \Im(\phi_1 \overline{\phi_2}))}{\|\Psi'\|^2}.
\]

At this juncture, we need the important map, which we shall denote by \texttt{stereo}, defined by the stereographic projection shown in Figure 22.2. It can be used to transfer information between the sphere \( S^2(1) \) of unit radius in \( \mathbb{R}^3 \) centered at the origin and the complex plane. We already discussed the inverse of \texttt{stereo} in Section 12.2, but now we use complex variables.

Let \( p \in S^2(1) \) be any point other than the north pole \( n = (0,0,1) \). The line \( \ell \) through \( p \) and \( n \) meets the complex plane \( \mathbb{C} \) at a point \( q \). Let us identify \( \mathbb{C} \) with the equatorial plane \( \mathbb{R}^2 \). Write \( p = (p_1, p_2, p_3) \) and \( q = (q_1, q_2, 0) \). We can parametrize \( \ell \) as
\[
t \mapsto tq + (1-t)n = (tq_1, tq_2, 1-t).
\]
Thus there is a \( t_0 \) such that
\[
p = t_0q + (1-t_0)n = (t_0q_1, t_0q_2, 1-t_0).
\]
It follows that $t_0 = 1 - p_3$, so that
\[ q_1 = \frac{p_1}{t_0} = \frac{p_1}{1 - p_3} \quad \text{and} \quad q_2 = \frac{p_2}{t_0} = \frac{p_2}{1 - p_3}. \]

\[ \text{Figure 22.2: Stereographic projection from the north pole} \]

The construction is summarized by

**Definition 22.19.** Stereographic projection is the mapping $\text{stereo} : S^2(1) \setminus n \rightarrow \mathbb{C}$ given by
\[ \text{stereo}(p_1, p_2, p_3) = \frac{p_1 + ip_2}{1 - p_3}. \]

There is a simple formula for the composition of the stereographic projection and the Gauss map of a minimal curve in $\mathbb{C}^3$.

**Lemma 22.20.** Let $\Psi : U \rightarrow \mathbb{C}^3$ be a minimal curve, and write $\Psi' = (\phi_1, \phi_2, \phi_3)$. Let $U$ be the unit normal vector field of the patch given by (22.20). Then
\[ (22.23) \quad \text{stereo} \circ U = \frac{\phi_3}{\phi_1 - i\phi_2}. \]

**Proof.** Starting with Lemma 22.18, we see that $\text{stereo} \circ U$ equals
\[
\frac{2(\Im(\phi_2 \overline{\phi_3}) + i \Im(\phi_3 \overline{\phi_1}))}{\|\Psi'\|^2} = \frac{2(\Im(\phi_2 \overline{\phi_3}) + i \Im(\phi_3 \overline{\phi_1}))}{\|\Psi'\|^2 - 2 \Im(\phi_1 \overline{\phi_2})}
\]
\[ = \frac{\phi_2 \overline{\phi_3} - \phi_3 \phi_2 + i(\phi_3 \overline{\phi_1} - \overline{\phi_3} \phi_1)}{i(\|\Psi'\|^2 - 2 \Im(\phi_1 \overline{\phi_2}))}
\]
\[ = \frac{\phi_3(\phi_1 + i \phi_2) - \overline{\phi_3}(\phi_1 + i \phi_2)}{\|\Psi'\|^2 - 2 \Im(\phi_1 \overline{\phi_2})}. \]
Using the fact that $(\phi_1 - i \phi_2)(\phi_1 + i \phi_2) = -\phi_3^2$, we rewrite the right-hand side as
\[
\frac{\phi_3(\phi_1 + i \phi_2) + \overline{\phi_3} \left( \frac{\phi_3^2}{\phi_1 - i \phi_2} \right)}{\|\Psi'\|^2 - 2 \Im(\phi_1 \overline{\phi_2})} = \frac{\phi_3(\phi_1 + i \phi_2)(\phi_1 - i \phi_2) + |\phi_3|^2}{(\phi_1 - i \phi_2)(\|\Psi'\|^2 - 2 \Im(\phi_1 \overline{\phi_2}))} = \frac{\phi_3(|\Psi'|^2 + i \phi_2 \phi_1 - i \phi_2 \phi_1)}{(\phi_1 - i \phi_2)(\|\Psi'\|^2 - 2 \Im(\phi_1 \overline{\phi_2}))} = \frac{\phi_3}{\phi_1 - i \phi_2}.
\]

The version of Lemma 22.9 for a minimal curve is

**Lemma 22.21.** Let $\Psi : U \to \mathbb{C}^3$ be a minimal curve with unit normal vector field $U$ and associated family $t \mapsto z[t]$. Then the coefficients of the second fundamental form of $z[t]$ are given by
\[
\begin{align*}
e(t) &= -g(t) = \Re \left( (e^{-it}\Psi'') \cdot U \right), \\
f(t) &= -\Im \left( (e^{-it}\Psi'') \cdot U \right).
\end{align*}
\]

**Proof.** From (22.21), we get
\[
(e^{-it}\Psi'') \cdot U = (z[t]_{uu} - i z[t]_{uv}) \cdot U = e(t) - i f(t).
\]
and so (22.24) follows. 

Theorem 22.6 implies that all of the patches in the associated family of a minimal curve are isometric. Hence by Theorem 17.5 they all have the same Gaussian curvature, and we may speak of the **Gaussian curvature of a minimal curve.** We conclude this section by finding a formula for this quantity.

**Lemma 22.22.** The Gaussian curvature of a minimal curve $\Psi : U \to \mathbb{C}^n$ is given by
\[
K = -4\left(\frac{\|\Psi''\|^2 \|\Psi'\|^2 - |\Psi'' \cdot \overline{\Psi'}|^2}{\|\Psi'\|^6}\right).
\]

**Proof.** In the notation of Definition 16.14 on page 518, $\|\Psi'\|^2 = 2\lambda^2$, and so
\[
\frac{1}{2} \Delta \log(\lambda^2) = \frac{\partial^2}{\partial z \partial \overline{z}} \log (\|\Psi'\|^2) = \frac{\partial^2}{\partial z \partial \overline{z}} \log (\Psi' \cdot \overline{\Psi'}).
\]
The right-hand side equals
\[
\frac{\partial}{\partial \overline{z}} \left( \frac{\Psi'' \cdot \overline{\Psi'}}{\|\Psi'\|^2} \right) = \frac{\|\Psi'\|^2 \|\Psi''\|^2 - (\Psi'' \cdot \overline{\Psi'})(\Psi' \cdot \overline{\Psi'})}{\|\Psi'\|^4} = \frac{\|\Psi'\|^2 \|\Psi''\|^2 - |\Psi'' \cdot \overline{\Psi'}|^2}{\|\Psi'\|^4}.
\]
Then (17.11) on page 540 and (22.25) imply that
\[ K = -\frac{\Delta \log \lambda}{\lambda^2} = -\frac{2 \parallel \Psi' \parallel^2 (\parallel \Psi'' \parallel^2 - |\Psi'' \cdot \overline{\Psi}'|^2)}{\parallel \Psi' \parallel^4} \]
\[ = -\frac{4(\parallel \Psi' \parallel^2 \parallel \Psi'' \parallel^2 - |\Psi'' \cdot \overline{\Psi}'|^2)}{\parallel \Psi' \parallel^6}. \]

We now turn attention to the construction of minimal curves. This will be a two-step process. In the next section, we explain how a minimal isothermal patch gives rise to a minimal curve, and in the following one show how to write down minimal curves from scratch.

### 22.4 Finding Conjugate Minimal Surfaces

We know that a minimal isothermal patch \( x: U \to \mathbb{R}^n \) is the real part of an \( n \)-tuple \( (\psi_1, \ldots, \psi_n) \) of complex analytic functions. How do we explicitly determine \( (\psi_1, \ldots, \psi_n) \) from \( x \)?

First, let us ask how to determine a complex analytic function \( f: U \to \mathbb{C} \) such that the real part of \( f \) is a given harmonic function \( h: U \to \mathbb{R} \). Of course, \( f \) is unique only up to an imaginary constant. The standard way (found in most text books on complex variables) is to use the Cauchy–Riemann equations; this method requires integration. There is, however, a more efficient method (see [Ahlf, pages 27–28], [Boas, pages 158–159] and [Lait]) that is completely algebraic, and hence involves no integration whatever. The idea is to substitute complex variables formally for the real variables \( u \) and \( v \) in the expression \( h(u, v) \).

**Lemma 22.23.** Let \( h: U \to \mathbb{R} \) be a harmonic function where \( U \) is an open subset of \( \mathbb{R}^2 = \mathbb{C} \), and let \( z_0 = u_0 + iv_0 \in U \).

(i) Let \( f: U \to \mathbb{C} \) be a complex analytic function such that \( \Re f(u + iv) = h(u, v) \) and \( \Im f(z_0) = 0 \). Then
\[ f(z) = 2h\left( \frac{z + z_0}{2}, \frac{z - z_0}{2i} \right) - h(u_0, v_0). \]

(ii) Conversely, if \( f \) is defined by (22.26), then \( f: U \to \mathbb{C} \) is a complex analytic function such that \( f(z_0) = h(u_0, v_0) \).

**Proof.** For (i) we do the case \( z_0 = 0 \). Define \( g \) by \( g(z) = \overline{f(z)} \). Then \( g \) is complex analytic, and we can express \( h \) in terms of \( f \) and \( g \):
\[ h(u, v) = \frac{1}{2}(f(u + iv) + \overline{f(u + iv)}) = \frac{1}{2}(f(u + iv) + g(u - iv)). \]
Since $h$ is harmonic, it can be expanded in a double power series about any point $p \in U$. Thus, even though $z$ is a complex variable, we may calculate

$$h\left(\frac{z}{2}, \frac{z}{2i}\right) = \frac{1}{2}(f(z) + g(0)) = \frac{1}{2}(f(z) + h(0, 0)).$$

For (ii) we compute

$$f_u = h_u - ih_v \quad \text{and} \quad f_v = ih_u + h_v.$$ 

Therefore,

$$\frac{\partial f}{\partial \bar{z}} = 0,$$

and $f$ satisfies the Cauchy–Riemann equations (22.1).

Let $x: U \rightarrow \mathbb{R}^n$ be a minimal isothermal patch, and write

$$x(u, v) = (h_1(u, v), \ldots, h_n(u, v)).$$

Having fixed $(u_0, v_0) \in U$, we can apply Lemma 22.23 to each component $h_j$, so as to obtain a complex analytic function $\psi_j$ for which

(22.27) $h_j(u, v) = \Re \psi_j(u + iv) \quad \text{and} \quad 0 = \Im \psi_j(z_0).$

This allows us to record

**Definition 22.24.** The mapping $\Psi = (\psi_1, \ldots, \psi_n)$ whose components satisfy (22.27) is called a complexification of $x$.

The key point is that each component of $x$ is the real part of the corresponding component of $\Psi$. The ambiguity arising from this condition is resolved by demanding that $\Psi$ coincides with $x$ at $z_0$.

In the light of Lemma 22.23, the state of affairs can be summarized by

**Corollary 22.25.** Let $x: U \rightarrow \mathbb{R}^n$ be a minimal isothermal patch, where $U$ is an open subset of $\mathbb{R}^2 = \mathbb{C}$ containing $(0, 0)$. Let $\Psi = (\psi_1, \ldots, \psi_n)$ be the complexification of $x$ such that $\Im \psi(0) = 0$. Then

$$\Psi(z) = 2x\left(\frac{z}{2}, \frac{z}{2i}\right) - x(0, 0),$$

and the conjugate minimal isothermal patch $y$ of $x$ with $y(0, 0) = 0$ is given by

$$y(u, v) = \Im \left(2x\left(\frac{u + iv}{2}, \frac{u + iv}{2i}\right) - x(0, 0)\right).$$
Enneper’s Surface of Degree $n$

We can apply Corollary 22.25 with $x = \text{enneper}$, as defined on page 509. Replacing $u$ and $v$ by $z/2$ and $-iz/2$ respectively gives

\begin{equation}
\Psi(z) = \left(z - \frac{z^3}{3}, iz + \frac{z^3}{3}, z^2\right).
\end{equation}

This is then a minimal curve, and the conjugate to Enneper’s surface is

\begin{equation}
y(u, v) = \text{Im} \Psi(u + iv) = \left(v + \frac{u^3}{3} - u^2v, u + \frac{u^3}{3} - uv^2, 2uv\right).
\end{equation}

The associated family of \text{enneper} yields no essentially new surfaces because of the next result, which is verified in Notebook 22.

**Lemma 22.26.** Let $t \mapsto \text{enneper}[t]$ denote the associated family of Enneper’s surface. Then $\text{enneper}[t]$ coincides with a reparametrization of $\text{enneper}$ rotated around the $z$-axis. Hence there is a Euclidean motion of $\mathbb{R}^3$ that maps $\text{enneper}$ onto $\text{enneper}[t]$.

Formula (22.28) can be generalized by setting

\begin{equation}
\text{ennepermincurve}[n](z) = \left(z - \frac{z^{2n+1}}{2n+1}, iz + \frac{iz^{2n+1}}{2n+1}, \frac{2z^{n+1}}{n+1}\right).
\end{equation}

If we write

\begin{equation}
\text{ennepermincurve}[n](u, v) = (\phi_1, \phi_2, \phi_3),
\end{equation}

it is easy to check that

$$\phi_1^2 + \phi_2^2 + \phi_3^2 = 0,$$

and that $|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2$ is never zero. Theorem 22.13 now asserts that the real part of (22.29) is a minimal surface.

Like Enneper’s surface (the case $n = 1$), the patch

\begin{equation}
(u, v) \mapsto \text{Re}(\text{ennepermincurve}[n](u, v))
\end{equation}

is hard to visualize because of its many self-intersections. As an aid to understanding this surface in Notebook 22, we determine the polar parametrization of each surface in its associated family. Figure 22.3 displays the case of $n = 2$ (undeformed, so $t = 0$) with a maximum radius $r_{\text{max}}$ of 1.0, 1.6, 2.2 respectively. The curves on the right are the projections of the ‘polar boundary’ corresponding to $r_{\text{max}}$, and make the intersections self-evident.
Figure 22.3: Polar plots of the surface enneper\cite{2} with increasing radii
22.5 The Weierstrass Representation

The proof of the following result is almost obvious, yet it is of fundamental importance.

**Theorem 22.27.** Let $f$ and $g$ be complex analytic functions defined in a region $\mathcal{U}$ of the complex plane $\mathbb{C}$. Fix $z_0 \in \mathcal{U}$, and define $\Psi = (\psi_1, \psi_2, \psi_3): \mathcal{U} \to \mathbb{C}^3$ by

\[
\begin{aligned}
\psi_1(z) &= \int_{z_0}^{z} \frac{f(w)}{2} (1 - g(w)^2) \, dw, \\
\psi_2(z) &= \int_{z_0}^{z} \frac{i f(w)}{2} (1 + g(w)^2) \, dw, \\
\psi_3(z) &= \int_{z_0}^{z} f(w)g(w) \, dw.
\end{aligned}
\]

(22.30)

Then $\Psi$ is a minimal curve.

**Proof.** The complex derivative of $\Psi$ equals

\[
\left( \frac{1}{2} f(z)(1 - g(z)^2), \frac{i}{2} i f(z)(1 + g(z)^2), f(z)g(z) \right),
\]

and this is set up precisely to ensure that (22.19) is satisfied. $\blacksquare$

We call $\Psi$ the **Weierstrass minimal curve** starting at $z_0$ determined by $f(z)$ and $g(z)$. Put

\[
\begin{aligned}
x(u, v) &= \Re \left( \psi_1(u, v), \psi_2(u, v), \psi_3(u, v) \right), \\
y(u, v) &= \Im \left( \psi_1(u, v), \psi_2(u, v), \psi_3(u, v) \right),
\end{aligned}
\]

(22.32)

so that $\Psi$ is a complexification of $x$. Since each $\psi_j$ is complex analytic, we know that

\[x_{uu} + x_{vv} = y_{uu} + y_{vv} = 0,
\]

and $x, y$ are both minimal isothermal patches.

**Definition 22.28.** The map $x$ is called the **Weierstrass patch** and $y$ the conjugate **Weierstrass patch**, starting at $z_0$, determined by $f(z)$ and $g(z)$.

**Proposition 22.29.** The metric of both the Weierstrass patch and the conjugate Weierstrass patch is given by

\[
\begin{aligned}
ds^2 &= \frac{1}{4} \left| f(z) \right|^2 \left( 1 + \left| g(z) \right|^2 \right)^2 \, dz^2.
\end{aligned}
\]

In particular, $x$ and $y$ are isometric. They are regular except where $f$ is zero or singular.
Proof. From (22.16), page 726, we have

\[ E = G = 2 \sum_{k=1}^{3} |\phi_k[x]|^2 \]

\[ = \frac{1}{8} \left( |f|^2 |1 - g|^2 + |f|^2 |1 + g|^2 + 4|fg|^2 \right) = \frac{1}{4} |f|^2 (1 + |g|^2)^2. \]

The result follows since \( F = 0 \) and \( du^2 + dv^2 = |dz|^2 \).

As a first example, consider \( f(z) = -e^{-z} \) and \( g(z) = -e^z \).

Equation (22.31) yields

\[ \Psi'(z) = (i \sin z, -i \cos z, 1). \]

Taking \( z_0 = 0 \) gives

\[ \Psi(z) = (i(1 - \cos z), -i \sin z, z). \]

With the aid of the formulas

\[
\begin{align*}
\sin z &= \sin u \cosh v + i \cos u \sinh v, \\
\cos z &= \cos u \cosh v - i \sin u \sinh v,
\end{align*}
\]

it can be verified that the associated family coincides with the deformation from a helicoid to a catenoid; the resulting formulas are close to (16.8) on page 505.

Although we required the functions \( f, g \) above to be complex analytic, this is too restrictive a hypothesis in practice. Indeed, many of the well-known minimal surfaces can only be constructed using Weierstrass patches in which \( f \) and \( g \) are allowed to have isolated singularities. More precisely, we shall take \( f, g \) to be meromorphic functions, a concept we next recall.

Fix \( z_0 \in \mathbb{C}, a > 0, \) and let \( \mathcal{U} \) be the ‘punctured disk’ \( \{z \mid 0 < |z - z_0| < a\} \).

A complex analytic function \( f: \mathcal{U} \to \mathbb{C} \) is said to have a pole or order \( k \) at \( z_0 \) (where \( k \geq 1 \)) if

\[ \lim_{z \to z_0} (z - z_0)^k f(z) \]

exists, but \( |(z - z_0)^{k-1} f(z)| \to \infty \) as \( z \to z_0 \). Extending the value of a function to \( z_0 \) to make it continuous whenever possible, we may say that \( f \) has a pole of order \( k \) at \( z_0 \) if and only if \( 1/f \) has a zero or order \( k \), which is perhaps easier to imagine. A function \( f: \mathbb{C} \to \mathbb{C} \) is called meromorphic provided it is defined at all points apart from a (possibly infinite) number of poles.

A good example of the necessity to allow poles is provided by Henneberg’s minimal surface, page 511, which is given by
Lemma 22.30. The Weierstrass patch determined by the functions
\[ f(z) = 1 - \frac{1}{z^4} \quad \text{and} \quad g(z) = z \]
is a reparametrization of the surface \((u, v) \mapsto \frac{1}{2}\text{Henneberg}(u, -v)\).

The function \(f\) has a pole of order 4 at \(z_0 = 0\).

A converse to Theorem 22.27 also holds:

Lemma 22.31. Let \(\Psi : U \rightarrow \mathbb{C}^3\) be a minimal curve and write \(\Psi' = (\phi_1, \phi_2, \phi_3)\). Suppose that \(\phi_1 - i\phi_2\) is not identically zero. Define
\[ f = \phi_1 - i\phi_2, \quad \text{and} \quad g = \frac{\phi_3}{\phi_1 - i\phi_2}. \]
Then \(f\) and \(g\) give rise to the Weierstrass representation of \(\Psi\); that is,
\[ \Psi' = \left( \frac{1}{2}f(1 - g^2), \frac{1}{2}if(1 + g^2), fg \right). \]

Proof. It is easy to check that if \(f\) and \(g\) are defined by (22.34), then
\[ \phi_1 = \frac{1}{2}f(1 - g^2), \quad \phi_2 = \frac{1}{2}if(1 + g^2) \quad \text{and} \quad \phi_3 = fg. \]

The associated family of the Weierstrass patch determined by meromorphic functions \(f, g\) is given by substituting (22.32) into (22.3) or (22.21). In so doing, we immediately discover

Lemma 22.32. The patch \(z[t]\) is the Weierstrass patch determined by the meromorphic functions \(e^{-it}f\) and \(g\).

Let us specialize the result of Lemma 22.22 to a Weierstrass patch.

Theorem 22.33. The Gaussian curvature of a Weierstrass patch determined by meromorphic functions \(f\) and \(g\) is given by
\[ K = \frac{-16|g'|^2}{|f|^2(1 + |g|^2)^4}. \]

The same formula holds for the Gaussian curvature of each member of the associated family, including the conjugate Weierstrass patch determined by \(f, g\).

Proof. We shall follow the proof (rather than the statement) of Lemma 22.22. We have
\[ \Delta \log |f(z)| = 2 \frac{\partial^2}{\partial z \partial \bar{z}} (\log |f(z)|)^2 = 2 \frac{\partial^2}{\partial z \partial \bar{z}} (\log f(z) + \log \overline{f(z)}) = 0, \]
and
\[ \Delta \log(1 + |g(z)|^2) = 4 \frac{\partial^2}{\partial z \partial \bar{z}} \log(1 + g(z)\bar{g}(z)) \]
\[ = 4 \frac{\partial}{\partial z} \left( \frac{g'(z)\bar{g}(z)}{1 + |g(z)|^2} \right) = \frac{4|g'(z)|^2}{(1 + |g(z)|^2)^2}. \]

We know from Proposition 22.29 that the scaling factor \( \lambda \) equals \( |f|(1 + |g|)^2 \).

It therefore follows that
\[ \frac{1}{4} K = \frac{-\Delta \log(|f|(1 + |g|)^2)}{|f|^2(1 + |g|^2)^2} = \frac{-\Delta \log(1 + |g|^2)}{|f|^2(1 + |g|^2)^2} \]
\[ = \frac{-4|g'|^2}{|f|^2(1 + |g|^2)^2} = \frac{-4|g'|^2}{|f|^2(1 + |g|^2)^2}. \]

It follows that the Gaussian curvature of a Weierstrass patch determined by
meromorphic functions \( f, g \)
vanishes precisely at the zeros of \( g' \). Hence if \( g' \)
is not identically zero, the zeros of the Gaussian curvature are isolated. Note that the zeros of the Gaussian curvature of any minimal surface are both planar points and umbilic points.

There is a very simple formula for the Gauss map, or what amounts to the
same thing, the unit normal, of a Weierstrass patch.

**Theorem 22.34.** Let \( U \) be the unit normal of a Weierstrass patch determined
by meromorphic functions \( f, g \). Then \( \text{stereo} \circ U \), considered as a function of a
complex variable, is precisely the function \( g \). Furthermore,
\[
U(z) = \left( \frac{2 \Re g(z)}{|g(z)|^2 + 1}, \frac{2 \Im g(z)}{|g(z)|^2 + 1}, \frac{|g(z)|^2 - 1}{|g(z)|^2 + 1} \right).
\]

**Proof.** Lemma 22.20 on page 730 implies that
\[ \text{stereo} \circ U = \frac{\phi_3}{\phi_1 - i \phi_2} = \frac{fg}{f(1 - g^2)/2 + f(1 + g^2)/2} = g. \]

We also know (from page 371) that the inverse of stereographic projection is the function \( \mathbb{C} \to S^2(1) \) given by
\[ \text{stereo}^{-1}(w) = \Upsilon(w) = \left( \frac{2 \Re w}{|w|^2 + 1}, \frac{2 \Im w}{|w|^2 + 1}, \frac{|w|^2 - 1}{|w|^2 + 1} \right). \]

Formula (22.36) now follows. \( \blacksquare \)

We have already noted in Theorem 22.6 that the Gauss map is the same for all
members of an associated family. So (22.36) is the formula for the unit normal
of any member of the associated family of a Weierstrass patch.
Richmond’s Minimal Surface

An interesting sequence of minimal surfaces more complicated than either the catenoid or Enneper’s surface is generated by the Weierstrass representation when

\[ f(z) = \frac{1}{z^2} \quad \text{and} \quad g(z) = z^{n+1}. \]

These surfaces have ‘one planar end’, in the sense that far away from a center, their Gaussian curvature \( K \) tends to zero, and in this region they resemble the plane. Figure 22.4 illustrates the case \( n = 1 \), which is named after Richmond\(^4\).

\[ \Psi[n](z) = -\frac{1}{2z} - \frac{z^{2n+1}}{4n+2}, \quad -\frac{i}{2z} + \frac{iz^{2n+1}}{4n+2}, \quad \frac{z^n}{n}. \]

Like (22.29), this can be studied effectively using polar coordinates, and we can define an associated family

\[ \rho[n][\theta](r, \theta) = \left( -\frac{\cos(t + \theta)}{2r} - \frac{r^{2n+1} \cos(t - (2n+1)\theta)}{4n+2}, \right. \]

\[ \left. -\frac{\sin(t + \theta)}{2r} + \frac{r^{2n+1} \sin(t - (2n+1)\theta)}{4n+2}, \quad r^n \cos(t - n\theta) \right). \]

Fortuitously, the Gaussian curvature \( K \) is a function of \( r \) alone, and approaches its supremum 0 as \( r \) tends to zero or infinity.

---

We return to Richmond’s surface by setting \( n = 1 \), and seek the minimum value of the Gaussian curvature of \( \rho[1] \). Calculations in Notebook 22 show that \( r = (3/5)^{1/4} \) is a critical point of \( K \), and it can be checked (for example, by plotting \( K \) as a function of \( r \)) that this is an absolute minimum.

### 22.6 Minimal Surfaces via Björling’s Formula

In 1844, Björling\(^5\) asked if it is possible to find a minimal surface containing a given analytic strip [Bj]. This question is frequently called Björling’s problem. H. A. Schwarz\(^6\) (see [Schwa, pages 179–189]) gave an elegant solution to it. For refinements of Schwarz’s method see [Nits, pages 139–145] and [DHKW, pages 120–135]. We first make this notion precise.

Let \( \alpha : (a, b) \to \mathbb{R}^3 \) be a curve. By a **holomorphic extension** of \( \alpha \), we mean a complex analytic mapping \( \tilde{\alpha} : \mathbb{R} \to \mathbb{C}^3 \) defined on a rectangle

\[
\mathcal{R} = \{ u + iv : u \in (a, b), v \in (c, d) \} \quad \text{with} \quad c < 0 < d,
\]

and \( \tilde{\alpha}(u) = \alpha(u) \) for all \( u \in (a, b) \). Such an extension is unique if it exists (in which case \( \alpha \) is itself **real analytic**), and so we can safely omit the tilde and use the same symbol \( \alpha \) for the extension. We shall also denote \( \mathcal{R} \) simply by \((a, b) \times (c, d)\).

**Definition 22.35.** Let \( \alpha, \gamma : (a, b) \to \mathbb{R}^3 \) be curves such that

\[
\| \gamma \| = 1 \quad \text{and} \quad \alpha' \cdot \gamma = 0.
\]

Suppose that there exist holomorphic extensions \( \alpha, \gamma : (a, b) \times (c, d) \to \mathbb{C}^3 \), such that (22.37) holds also for \( z \in (a, b) \times (c, d) \). Fix \( z_0 \in (a, b) \times (c, d) \). Then the **Björling curve** corresponding to \( \alpha \) and \( \gamma \) is given by

\[
\text{bjoerling}_{\alpha, \gamma}(z) = \alpha(z) - i \int_{z_0}^{z} \gamma(z) \times \alpha'(z) \, dz.
\]

We now prove that a Björling curve is in fact a minimal curve, and we show how it is related to \( \alpha \) and \( \gamma \).

**Theorem 22.36.** With the same hypotheses as Definition 22.35, we have

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\(^5\)Emanuel Gabriel Björling (1808–1872). Swedish mathematician. Professor at the University of Uppsala.

\(^6\)Herman Amandus Schwarz (1843–1921). German mathematician, who made fundamental contributions to the theories of minimal surfaces and conformal mappings. His work on eigenvalues lead to what we now call the Cauchy-Schwarz inequality.
(i) the mapping \( z \mapsto \text{bjorling}[\alpha, \gamma](z) \) is a minimal curve;

(ii) for real \( u \), the map \( u \mapsto \Re(\text{bjorling}[\alpha, \gamma](u)) - \alpha(u) \) is constant;

(iii) for real \( u \), \( U(u, 0) = \gamma(u) \), where \( U \) denotes the unit normal to the patch \( (u, v) \mapsto \Re(\text{bjorling}[\alpha, \gamma](u + iv)) \);

**Proof.** Write \( \Psi = \text{bjorling}[\alpha, \gamma] \); then \( \Psi' = \alpha' - i\gamma \times \alpha' \). Using (7.7), page 195, and (22.37), we find that

\[
\Psi' \cdot \Psi' = (\alpha' - i\gamma \times \alpha') \cdot (\alpha' - i\gamma \times \alpha') \\
= \alpha' \cdot \alpha' - (\gamma \times \alpha') \cdot (\gamma \times \alpha') - 2i \alpha' \cdot (\gamma \times \alpha') \\
= \alpha' \cdot \alpha' - (\gamma \cdot \gamma)(\alpha' \cdot \alpha') = 0.
\]

Hence \( \Psi \) is a minimal curve.

Now, (ii) holds because \( \alpha(u) \) and \( \gamma(u) \) are real whenever \( u \) is itself real. To prove (iii), we observe that

\[
\Psi'(u) = \alpha'(u) - i\gamma(u) \times \alpha'(u) \quad \text{and} \quad \overline{\Psi'(u)} = \alpha'(u) + i\gamma(u) \times \alpha'(u)
\]

for \( u \in \mathbb{R} \). Hence by (22.37),

\[
\Psi'(u) \cdot \overline{\Psi'(u)} = 2\alpha'(u) \cdot \alpha'(u),
\]

and by (7.3) on page 193,

\[
\Psi'(u) \times \overline{\Psi'(u)} = 2i \alpha'(u) \times (\gamma(u) \times \alpha'(u)) = 2i(\alpha'(u) \cdot \alpha'(u))\gamma(u)
\]

Using Lemma 22.18, we find that

\[
U(u, 0) = \frac{\Psi'(u) \times \overline{\Psi'(u)}}{i \|\Psi'(u)\|^2} = \gamma(u).
\]

We can use Theorem 22.36 to construct many interesting minimal surfaces. For example, let \( \alpha: (a, b) \to \mathbb{R}^2 \) be a regular plane curve which has a regular holomorphic extension \( \alpha: (a, b) \times (c, d) \to \mathbb{C}^2 \). We take \( \gamma = J\alpha' / \|\alpha'\| \). Then

\[
\gamma \times \alpha' = -\|\alpha'\| E_3,
\]

where \( E_3 \) is the unit vector field perpendicular to \( \mathbb{R}^2 \) in \( \mathbb{R}^3 \). Thus

\[
\text{bjorling}[\alpha, \gamma](z) = \alpha(z) + i \left( \int_{z_0}^z \sqrt{\alpha'(z) \cdot \alpha'(z)} \, dz \right) E_3.
\]

This leads to the following definition.
Definition 22.37. Let $\alpha: (a, b) \to \mathbb{R}^2$ be a plane curve, and suppose that $\alpha$ has a holomorphic extension $\alpha: (a, b) \times (c, d) \to \mathbb{C}^2$. Write $\alpha = (a_1, a_2)$. We define

$$\text{bjmincurve}[\alpha](z) = \left( a_1(z), a_2(z), i \int_{a_1(z)}^{b_1(z)} \sqrt{a_1^2(z)^2 + a_2^2(z)^2} \, dz \right).$$

We need a result about the intersection of a plane and a surface in $\mathbb{R}^3$.

Theorem 22.38. Let $\beta$ be a unit-speed curve which lies on the intersection of a regular surface $M \subset \mathbb{R}^3$ and a plane $\Pi$. Suppose that $M$ meets $\Pi$ perpendicularly along $\beta$. Then $\beta$ is a geodesic in $\mathcal{M}$.

Proof. Let $U$ denote the unit normal vector field to $\mathcal{M}$. Then $\beta'$, $\beta''$ and $U \circ \beta$ all lie in $\Pi$, and $\beta'$ is perpendicular to $U \circ \beta$. Since $\beta$ is a unit-speed curve, $\beta''$ is perpendicular to $\beta'$. Hence $\beta''$ is a multiple of $U \circ \beta$, and so $\beta$ is a geodesic in $\mathcal{M}$.

Theorem 22.39. Let $\beta: (a, b) \to \mathbb{R}^2$ be a unit-speed plane curve which has a holomorphic extension $\beta: (a, b) \times (c, d) \to \mathbb{C}^2$, where $c < 0 < d$. Then

$$\Re(\text{bjmincurve}[\beta]): (a, b) \times (c, d) \to \mathbb{R}^3$$

is a minimal surface which contains $\beta$ as a geodesic.

Proof. Writing $\Psi = \text{bjmincurve}[\beta]$, it is immediate that

$$\Psi'[\beta] \cdot \Psi'[\beta] = 0,$$

so that $\Psi[\beta]$ really is a minimal curve. Furthermore, if we put $\beta = (b_1, b_2, b_3)$, then Lemma 22.18 expresses the unit normal of $\Psi[\beta]$ as

$$U = \frac{2 \left( \Im(-ib_2 \sqrt{b_1^2 + b_2^2}), \Im(ib_1 \sqrt{b_1^2 + b_2^2}), \Im(b_1 b_2) \right)}{|b_1|^2 + |b_2|^2 + |\sqrt{b_1^2 + b_2^2}|^2}.$$

Since $b_1'(z), b_2'(z)$ are real for $z \in \mathbb{R}$, we get

$$U(u, 0) = \frac{2 \left( -b_2 \sqrt{b_1^2 + b_2^2}, (b_1' \sqrt{b_1^2 + b_2^2}), 0 \right)}{2(b_1^2 + b_2^2)} \bigg|_{z=(u,0)} = \frac{(-b_2, b_1', 0)}{\sqrt{b_1^2 + b_2^2}} \bigg|_{z=(u,0)}.$$

Therefore, $U$ is perpendicular to $\mathbf{E}_3$, the unit normal of the $xy$-plane $\Pi$. Now Theorem 22.38 implies that $\beta$ is a geodesic in $\Re \Psi[\beta]$.

Corollary 22.40. Let $\alpha: (a, b) \to \mathbb{R}^2$ be a regular plane curve which has a holomorphic extension $\alpha: (a, b) \times (c, d) \to \mathbb{C}^2$, where $c < 0 < d$. Then

$$\Re(\text{bjmincurve}[\alpha]): (a, b) \times (c, d) \to \mathbb{R}^3$$

is a minimal curve which contains $\alpha$ as a pregeodesic.
Proof. Write $\Psi = \text{bjmincurve}[\beta]$ again. Then there exists an increasing analytic function $h: (c, d) \to (e, f)$ such that $\beta = \alpha \circ h$ is a unit-speed plane curve with a holomorphic extension. Since $\Psi[\beta](z) = \Psi[\alpha](h(z))$ and

$$\Psi[\beta]'(z) = \Psi[\alpha](h(z)) h'(z),$$

it follows that $\Psi[\beta]$ is a minimal curve with the same range as $\alpha$. Since $\beta$ is a geodesic in $\Re \Psi[\beta] = \Re \Psi[\alpha]$, we conclude that $\alpha$ is a pregeodesic in $\Re \Psi[\alpha]$.

Figure 22.5: A minimal surface with an ellipse as geodesic

As a first example, let us find a minimal surface that contains the cycloid

$$t \mapsto (t - \sin t, 1 - \cos t)$$

as a geodesic. The minimal curve obtained from Definition 22.37 is

$$\left( z - \sin z, 1 - \cos z, -4i \cos \frac{z}{2} \right).$$

To take its real part we use (22.33), and deduce that the parametrization of the corresponding minimal surface is

$$\left( u - \cosh v \sin u, 1 - \cos u \cosh v, -4 \sin \frac{u}{2} \sinh \frac{v}{2} \right).$$

Not surprisingly, the resulting minimal surface is precisely Catalan’s, given its description on page 510.
It is clear that a catenoid contains a circle as a geodesic, but what is the minimal surface that contains an ellipse as a geodesic? (See [Schwa, page 182].) We can solve this problem using the above methods, by first using the formula

\[ s(t) = b \, E(t \mid 1 - \frac{a^2}{b^2}) \]

for the arc length of the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \). This allows us to define the corresponding minimal curve

\[ \Phi(z) = \left( a \cos z, b \sin z, ib \, E(z \mid 1 - \frac{a^2}{b^2}) \right) \]

A computation from Notebook 22 then produces the associated family, and Figure 22.5 plots the minimal surface

\[ x(u, v) = \left( 2 \cos u \cosh v, \cosh v \sin u, -\text{Im} E(u + iv \mid -3) \right) \],

corresponding to \( a = 2 \), \( b = 1 \) and \( t = 0 \).

Many more examples are investigated in Notebook 22; see Exercise 13 and Figure 22.11 on page 754. We conclude this section by illustrating the minimal surface generated by one of the most elegant of all plane curves, namely the clothoid. This gives rise to a minimal curve

\[ \Phi(z) = \left( \sqrt{\pi} \text{FresnelS} \left( \frac{z}{\sqrt{\pi}} \right), \sqrt{\pi} \text{FresnelC} \left( \frac{z}{\sqrt{\pi}} \right), iz \right) \],

and the surface in Figure 22.6 is (a rotated version of) the member of the associated family with \( t = \pi/4 \). Since \( t \neq 0 \), this surface no longer contains the initial clothoid; as \( t \) tends to \( \pi/2 \) the corresponding surface resembles more and more a helicoid.
22.7 Costa’s Minimal Surface

One of the most interesting minimal surfaces was defined by Costa in 1984 (see [Costa1]). Hoffman and Meeks (see [HoMe]) showed that Costa’s minimal surface is a complete embedded surface. The term ‘complete’ will be properly defined on page 888, whilst ‘embedded’ means that the surface has no self-intersections. We shall adapt the treatment of Costa’s minimal surface given in [BaCo] to find a parametrization that can be investigated in Notebook 22. For this, we need the Weierstrass zeta function \( \zeta \). First, we explain the relevant theory of this function and also the Weierstrass function \( \wp \).

![Figure 22.7: Costa’s minimal surface](image)

The Weierstrass \( \wp \) function is a meromorphic function on the complex plane that is used to relate the 2-dimensional torus (as defined by Figure 11.4 on page 337) to complex cubic curves. The resulting introductory theory of elliptic curves is clearly presented in many texts, for example [Kir]. The function \( \wp \) is usually defined for an arbitrary lattice, meaning a set

\[
\{ m\omega_1 + n\omega_2 \mid m, n \in \mathbb{Z} \},
\]

where \( \omega_1, \omega_2 \) are complex numbers with \( \Im(\omega_2/\omega_1) \neq 0 \). There is little loss in generality in assuming that \( \omega_1 = 1 \), but we shall mainly consider the special case in which \( \omega_1 = 1 \) and \( \omega_2 = i \), so that our lattice becomes the set

\[
\mathcal{L} = \{ m + in \mid m, n \in \mathbb{Z} \}
\]

(22.38)

of Gaussian integers. It is also convenient to denote by \( \mathcal{L}^* \) the set \( \mathcal{L} \setminus \{0\} \) of non-zero lattice points.
Having defined $L^*$, we may define the Weierstrass $\wp$ function by
\begin{equation}
\wp(z) = \frac{1}{z^2} + \sum_{\omega \in L^*} \left( \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right).
\end{equation}

Then $\wp$ is defined for all $z \in \mathbb{C} \setminus L^*$, but has a pole of order 2 at each lattice point. Moreover, $\wp$ is known to be **doubly periodic**, in the sense that
\[ \wp(z + m + in) = \wp(z) \]
for all $m, n \in \mathbb{Z}$ and $z \in \mathbb{C} \setminus L^*$. Thus, $\wp$ is completely determined by its values on the **fundamental domain**
\[ \mathcal{F} = \{ z \in \mathbb{C} \mid z = \alpha + \beta i \text{ with } 0 \leq \alpha, \beta < 1 \}. \]

The Weierstrass $\wp$ function satisfies an addition formula:
\begin{equation}
\wp(z_1 + z_2) = \frac{1}{4} \left( \frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)} \right)^2 - \wp(z_1) - \wp(z_2).
\end{equation}

The next result establishes the link with cubic equations.

**Lemma 22.41.** The function $\wp$ is a solution of the differential equation
\begin{equation}
\wp'(z)^2 = (4\wp(z)^2 - c)\wp(z),
\end{equation}
where $c$ is a nonzero constant.

**Proof.** We shall deduce (22.41) from a more general, but well-known, result. Namely,
\[ \wp'(z)^2 = 4\wp(z)^3 - g_2 \wp(z) - g_3, \]
where
\begin{equation}
g_2 = 60 \sum_{\omega \in L^*} \frac{1}{\omega^4}, \quad \text{and} \quad g_3 = 140 \sum_{\omega \in L^*} \frac{1}{\omega^6}.
\end{equation}

This applies for a general lattice $L$ (see, for example, [Chand, page 29]), though we claim that in our case (22.38), $g_3 = 0$. To verify this, observe that
\[ (m - in)^6 + (m + in)^6 + (n - im)^6 + (n + im)^6 = 0; \]
it follows that the summands representing $g_3$ in (22.42) with $m^2 + n^2$ fixed cancel out. To obtain (22.41), we set $c = g_2$, and use Notebook 22 to obtain the approximation $c = 189.073$. \[ \square \]
Since \( \wp' \) is an odd function of \( z \) with period 1, we see that
\[
\wp'(\frac{1}{2}) = -\wp'(-\frac{1}{2}) = -\wp'(\frac{1}{2})
\]
is zero. Similarly,
\[
\wp'(\frac{i}{2}) = \wp'(\frac{1+i}{2}) = 0.
\]
Moreover, \( \wp' \) has no other zeros in \( \mathcal{F} \) besides \( \frac{1}{2}, \frac{i}{2}, (1+i)/2 \). It follows from (22.41) that these three numbers are the roots of the cubic polynomial
\[
(4\wp(z)^2 - c)\wp(z) = 0,
\]
and it is known that \( \wp((1+i)/2) = 0 \). It follows that
\[
e = \wp(\frac{1}{2}) = -\wp(\frac{i}{2})
\]
satisfies \( c = 4e^2 \), and (22.41) becomes
\[
\wp'(z)^2 = 4\wp(z)(\wp(z)^2 - e^2)
\]
(22.43)

We shall also need the Weierstrass \( \zeta \) function, which is defined by
\[
\zeta(z) = \frac{1}{z} + \sum_{\omega \in \mathcal{L}^*} \left( \frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right).
\]
(22.44)

It is easy to see from (22.39) that
\[
\zeta'(z) = -\wp(z).
\]
(22.45)

The function \( \zeta \) is not periodic, but it is known that
\[
\zeta(z + m + in) = \zeta(z) + 2m\zeta(\frac{1}{2}) + 2n\zeta(\frac{i}{2}),
\]
(22.46)

for integers \( m, n \).

**Lemma 22.42.** For all \( z \), we have
\[
\wp(z - \frac{1}{2}) - \wp(z - \frac{i}{2}) - 2e = \frac{16e^3\wp(z)}{\wp'(z)^2},
\]
(22.47)
\[
i\zeta(iz) = \zeta(z),
\]
(22.48)
\[
\zeta(\frac{1}{2}) = i\zeta(\frac{i}{2}) = \frac{\pi}{2},
\]
(22.49)
\[
\zeta(\frac{1+i}{2}) = \frac{(1-i)\pi}{2}.
\]
(22.50)

**Proof.** To verify (22.47), one uses (22.40) and (22.43) to prove
\[
\wp(z - \frac{1}{2}) = e + \frac{2e^2}{\wp(z) - e},
\]
and an analogous formula for \( \varphi(z - i/2) \). Equation (22.48) follows from the definition (22.44), and the second equality in (22.49) from Legendre's relation in the form

\[
\zeta\left(\frac{1}{2}\right) - \zeta\left(\frac{i}{2}\right) = i\pi.
\]

Finally, (22.50) follows from the formula

\[
\zeta\left(\frac{1+i}{2}\right) = \zeta\left(\frac{1}{2}\right) + \zeta\left(\frac{i}{2}\right),
\]

itself a consequence of (22.46). For more details, see [Chand, page 50].

A description of Costa's minimal surface using the Weierstrass function \( \wp \) has been given by Barbosa and Colares (see [BaCo, pages 84–90], and also [HoMe]). We adapt this description, but we use instead the Weierstrass zeta function \( \zeta \) to avoid one numerical integration. The use of \( \zeta \) instead of \( \wp \) speeds up the plotting of Costa's minimal surface by a factor of at least 50.

Costa's minimal surface can be defined as a Weierstrass patch using the functions

\[
f(z) = \varphi(z) \quad \text{and} \quad g(z) = \frac{A}{\wp'(z)}.
\]

In order that Costa's minimal surface have no self-intersections, we need to take

\[
A = \sqrt{2\pi c} = 2\sqrt{2\pi e} \approx 34.46707
\]

(see [BaCo, page 89]). The Costa minimal curve is then the minimal curve \( \Psi : \mathbb{C} \to \mathbb{C}^3 \) defined as follows. We put

\[
2\Psi'(z) = \left( \frac{1}{2} f(z)(1 - g(z)^2), \ \frac{i}{2} f(z)(1 + g(z)^2), \ f(z)g(z) \right),
\]

and take \( \Psi \) to be the antiderivative of \( \Psi' \) with the normalization \( \Psi((1+i)/2) = 0 \).

Next, we show how to use the Weierstrass function \( \zeta \) to express \( \Psi \) without integrals.

**Theorem 22.43.** The Costa minimal curve is given by \( \Psi = (\psi_1, \psi_2, \psi_3) \) where

\[
\begin{align*}
\psi_1 &= \frac{1}{2} \left\{ -\zeta(z) + \pi z - i\pi + \frac{\pi^2(1 + i)}{4e} \right. \\
&\quad \left. + \frac{\pi}{2e} \left[ \zeta(z - \frac{1}{2}) - \zeta(z - \frac{i}{2}) \right] \right\}, \\
\psi_2 &= \frac{i}{2} \left\{ -\zeta(z) - \pi z + \pi - \frac{\pi^2(1 + i)}{4e} \\
&\quad - \frac{\pi}{2e} \left[ \zeta(z - \frac{1}{2}) - \zeta(z - \frac{i}{2}) \right] \right\}, \\
\psi_3 &= \sqrt{2\pi} \left\{ \log \left( \frac{\varphi(z) - e}{\varphi(z) + e} \right) - \pi i \right\}.
\end{align*}
\]
Proof. We shall only prove the expression for \( \psi_1 \). Using (22.47) and (22.53), we obtain

\[
f(w)\left(1 - g(w)^2\right) = \varphi(w) - \frac{A^2 \varphi(w)}{\varphi'(w)^2}
\]

\[
= \varphi(w) - \frac{A^2}{16e^3} \left( \varphi(w - \frac{1}{2}) - \varphi\left(w - \frac{i}{2}\right) - 2e \right)
\]

\[
= \varphi(w) - \frac{7\pi}{2e} \left( \varphi(w - \frac{1}{2}) - \varphi\left(w - \frac{i}{2}\right) - 2e \right)
\]

\[
= \varphi(w) + \pi - \frac{e}{2e} \left( \varphi\left(w - \frac{1}{2}\right) - \varphi\left(w - \frac{i}{2}\right) \right).
\]

Integrating both sides and using (22.49) and (22.50), we get

\[
\int_{z/(1+i)/2}^z f(w)\left(1 - g(w)^2\right) \, dw
\]

\[
= \left\{ -\zeta(w) + \pi w + \frac{\pi}{2e} \left( \zeta\left(w - \frac{1}{2}\right) - \zeta\left(w - \frac{i}{2}\right) \right) \right\} \bigg|_{(1+i)/2}^z
\]

\[
= -\zeta(z) + \pi z + \frac{\pi}{2e} \left( \zeta\left(z - \frac{1}{2}\right) - \zeta\left(z - \frac{i}{2}\right) \right)
\]

\[
+ \zeta\left(\frac{1+i}{2}\right) - \frac{\pi(1+i)}{2e} - \frac{\pi}{2e} \left( \zeta\left(\frac{i}{2}\right) - \zeta\left(\frac{1}{2}\right) \right)
\]

\[
= -\zeta(z) + \pi z + \frac{\pi}{2e} \left( \zeta\left(z - \frac{1}{2}\right) - \zeta\left(z - \frac{i}{2}\right) \right) - i\pi + \frac{\pi^2(1+i)}{4e}.
\]

The formulas for \( \psi_2 \) and \( \psi_3 \) are similar. \( \blacksquare \)

*Figure 22.8: The lower part of Costa’s minimal surface*
Corollary 22.44. Costa’s minimal surface is given by \((x_1, x_2, x_3)\) where

\[
\begin{align*}
  x_1(u,v) &= \frac{1}{2} \Re \left\{ -\zeta(u+iv) + \pi u + \frac{\pi^2}{4e} ight. \\
  & \quad + \frac{\pi}{2e} \left[ \zeta(u+iv - \frac{1}{2}) - \zeta(u+iv + \frac{1}{2}) \right] \}, \\
  x_2(u,v) &= \frac{1}{2} \Re \left\{ -i\zeta(u+iv) + \pi v + \frac{\pi^2}{4e} ight. \\
  & \quad - \frac{\pi i}{2e} \left[ \zeta(u+iv - \frac{1}{2}) - \zeta(u+iv + \frac{1}{2}) \right] \}, \\
  x_3(u,v) &= \frac{\sqrt{2\pi}}{4} \log \left| \frac{\varphi(u+iv) - e}{\varphi(u+iv) + e} \right|. 
\end{align*}
\]

Further technical details regarding the above parametrization can be found in [FeGrMa]. It was used by Ferguson to craft sculptures of Costa’s surface in various materials, including snow. He produced a limited edition of fifty miniature bronze sculptures of the surface for the international conference in memory of the author in Bilbao in 2000; the inscription of each includes the expression ‘\(D_4\)’ indicating the symmetry group that Costa’s surface possesses.

Figure 22.9: Costa’s minimal surface seen from above

Helaman Ferguson trained as a stone mason and artist before embarking on a university career in mathematics. He now combines both disciplines to produce world-renowned sculptures, many of which represent sophisticated surfaces in differential geometry [Ferg]. He also designs algorithms for operating machinery and for scientific visualization.
22.8 Exercises

1. Give details for the proofs of Lemma 22.5, Corollary 22.11 and Lemma 7.3.

2. Show that if \( \Psi: \mathcal{U} \to \mathbb{C}^n \) is a minimal curve, then
\[
\Psi' \cdot \Psi'' = 0 \quad \text{and} \quad \Psi' \cdot \Psi''' + \Psi'' \cdot \Psi'' = 0.
\]

3. Let \( \Psi: \mathcal{U} \to \mathbb{C}^3 \) be a minimal curve such that \( \Psi' \) and \( \Psi'' \cdot \Psi'' \) never vanish.
Show that \( \Psi', \Psi'', \Psi''' \) are linearly independent at all points of \( \mathcal{U} \).

4. Let \( \Psi: \mathcal{U} \to \mathbb{C}^3 \) be a minimal curve such that \( \Psi' \) and \( \Psi'' \cdot \Psi'' \) never vanish.
Show that
\[
\Psi' \times \Psi'' = -\left( \frac{\Psi' \times \Psi'' \cdot \Psi''}{\Psi'' \cdot \Psi''} \right) \Psi'.
\]

5. Let \( m \geq 2 \) be an integer. Define Bour's minimal curve by
\[
\Phi(z) = \left( \frac{z^m - 1}{m - 1} \right) - \frac{z^{m+1}}{m + 1}, \quad i \left( z^m + \frac{z^{m+1}}{m + 1} \right) - \frac{2z^m}{m}
\]
Show that \( \Phi \) coincides with Enneper's minimal surface when \( m = 2 \). Find an explicit expression of the polar parametrization
\[
(r, \theta) \mapsto z[t](r \cos \theta, r \sin \theta)
\]
of each member of the associated family, and compute its Gaussian curvature. Plot the surface \( z[0] \) for \( m = 3 \).

6. Let \( x: \mathcal{U} \to \mathbb{R}^3 \) be any patch. Show that
\[
\frac{\partial}{\partial \bar{z}} \left( \frac{e - g^2}{2} - i f \right) = \lambda \frac{\partial H}{\partial z}.
\]
Conclude that for a surface of constant mean curvature the function
\[
u + iv \mapsto \frac{e(u, v) - g(u, v)}{2} - i f(u, v)
\]
is complex analytic.

7. Plot the Weierstrass patch for which \( f(z) = 2 \) and \( g(z) = z^{-1} + z^3 \). The bottom of the resulting surface should resemble a catenoid and the top should resemble Enneper’s minimal surface.

---

EDMOND BOUR (1832–1866). French mathematician, who made significant contributions to analysis, algebra, geometry and applied mechanics despite his early death.
8. The trinoid is the Weierstrass patch determined by $z \mapsto 1/(z^3 - 1)^2$ and $z \mapsto z^2$. Plot the trinoid using polar coordinates and compute its Gaussian curvature.

Figure 22.10: Two trinoide views

9. Show that for a Weierstrass minimal curve $\Psi$ whose derivative is given by
\[
\Psi' = \left(\frac{1}{2}f(1 - g^2), \frac{1}{2}if(1 + g^2), fg\right),
\]
we have the formula
\[
(22.55) \quad \Psi' \cdot U = -fg',
\]
where $U$ denotes the unit normal to $\Psi$.

10. Prove the following facts about the Gauss map of minimal surfaces:
    
    (a) The Gauss map of the catenoid is one-to-one; its image covers the sphere except for two points.
    
    (b) The image of the Gauss map of the helicoid omits exactly two points; otherwise, the Gauss map sends infinitely many points onto each point in its image.
    
    (c) The image of the Gauss map of Scherk’s surface omits exactly four points; otherwise, the Gauss map maps infinitely many points onto each point in its image.

11. Let $x: \mathcal{U} \to \mathbb{R}^n$ be a patch. The mean curvature vector field $H$ of $x$ is defined to be the component of
\[
\frac{Gx_{uu} - 2Fx_{uv} + Ex_{vv}}{2(EG - F^2)}
\]
that is perpendicular to \( x_u \) and \( x_v \). Show that this vanishes if and only if \( x \) is both isothermal and harmonic.

12. Show that the conjugate of Henneberg’s surface is given by

\[
y(u, v) = \left(2 \cosh u \sin v - \frac{2}{3} \cosh 3u \sin 3v, \\
2 \cosh u \cos v + \frac{2}{3} \cosh 3u \cos 3v, \\
2 \sinh 2u \sin 2v\right),
\]

as stated on page 512.

13. Construct a minimal surface that contains a hyperbola as a geodesic.

14. Investigate the minimal surface shown in Figure 22.10, arising from the minimal curve

\[
\Psi(z) = \left(\frac{a \cos z}{1 + \sin^2 z}, \frac{a \cos z \sin z}{1 + \sin^2 z}, i a F(z, -1)\right),
\]

where the elliptic function \( F \) is the inverse of Jacobi function \( \text{am} \), so that \( u = F(\phi | m) \) if and only if \( \phi = \text{am}(u | m) \). The surface contains a lemniscate as a geodesic.

\[\text{Figure 22.11: A minimal surface with a lemniscate as geodesic}\]

15. Draw a minimal surface that contains a cardioid as a geodesic.