Chapter 24

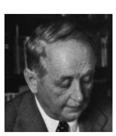
Differentiable Manifolds

We have reached a stage for which it is beneficial to take on board the theory of manifolds. This more abstract theory is a vehicle in which the definitions of Chapter 9 for Euclidean spaces extend in a natural way. The resulting concepts will provide us with a framework in which to pursue the intrinsic study of surfaces begun in Chapter 17.

The modern concept of a differentiable manifold, due to Weyl[1] (see [Weyl]), did not appear until the early twentieth century. A *manifold*, generally speaking, is a topological space which resembles Euclidean space locally. A *differentiable manifold* is a manifold $\mathcal{M}$ for which this resemblance is sharp enough to allow partial differentiation and consequently all the features of differential calculus on $\mathcal{M}$. The study of differentiable manifolds involves topology, since differentiability implies continuity, but metric properties of Euclidean space are not *a priori* included. In this chapter, we define the notion of differentiable manifold and some of the standard apparatus associated with it. Metrics and distances are discussed in the following two chapters.

The set $\mathbb{R}^n$ of n-tuples of real numbers is not only a vector space, but also a topological space, and the vector operations are continuous with respect to the topology. In addition, there is the notion of differentiability of real-valued functions on $\mathbb{R}^n$: we say that $f\colon \mathbb{R}^n \to \mathbb{R}$ is *differentiable* provided that the partial derivatives

$$\frac{\partial^{i_1+\cdots+i_r} f}{\partial u_1^{i_1} \ldots \partial u_r^{i_r}}$$

1  Hermann Klaus Hugo Weyl (1885–1955). German mathematician, who together with É. Cartan created the basis for the modern theory of Lie groups and their representations. With his application of group theory to quantum mechanics, he set up the modern theory of the subject. He was professor in Zürich and Göttingen. From 1933 until he retired in 1952, he worked at the Institute of Advanced Study at Princeton.

of all orders exist. Such functions are called C^∞ *functions*. They contrast with, on the one hand, the C^k *functions* (the functions which have continuous partial derivatives whenever $i_1 + \cdots + i_r \leqslant k$) and, on the other hand, the *real analytic functions* (functions which have convergent power series). If we denote these respective classes by $C^\infty(\mathbb{R}^n)$, $C^k(\mathbb{R}^n)$, $C^\omega(\mathbb{R}^n)$, then we have the strict inclusions

$$C^\omega(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n) \subset C^k(\mathbb{R}^n).$$

We choose to base our definitions on the class of C^∞ functions; use of C^k requires too complicated an approach, and use of C^ω is too restrictive.

In Section 24.1, we give the definition of differentiable manifold and describe some simple examples. The n-dimensional sphere is made into a differentiable manifold by introducing maps that generalize the stereographic map studied in Sections 8.6 and 22.3. The algebra $\mathfrak{F}(\mathcal{M})$ of C^∞ functions on a differentiable manifold is described in Section 24.2. The ensuing definitions generalize those already given in Chapter 12 for functions on and between surfaces. The tangent space to a differentiable manifold is defined in Section 24.3, and derives from the analogous Section 9.3. Maps between differentiable manifolds are studied in Section 24.4, and vector fields and tensor fields in Sections 24.5 and 24.6.

We shall assume that the reader is familiar with all the basic information about calculus of several variables, including the implicit function theorem and existence theorems for ordinary differential equations. This material can be found in, for example, [Spiv].

24.1 The Definition of a Differentiable Manifold

Recall the definition of the ***natural coordinate functions*** of $\mathbb{R}^n$ given on page 273; these are the mappings $u_i\colon \mathbb{R}^n \to \mathbb{R}$ defined by

$$u_i(p_1, \dots, p_n) = p_i$$

for $i = 1, \dots, n$, and we shall resolutely distinguish the functions u_i and the numbers p_i. A function $\Psi\colon \mathbb{R}^m \to \mathbb{R}^n$ is ***differentiable***, ***continuous*** or ***linear*** if and only if each $u_i \circ \Psi$ is differentiable, continuous or linear, respectively. This is consistent with Definition 9.7 on page 268.

We now make precise the 'local resemblance' referred to on the previous page.

Definition 24.1. *A* ***patch*** *on a topological space* $\mathcal{M}$ *is a pair* $(\mathbf{x}, \mathcal{U})$*, where* $\mathcal{U}$ *is an open subset of* $\mathbb{R}^n$ *and*

$$\mathbf{x}\colon \mathcal{U} \longrightarrow \mathbf{x}(\mathcal{U}) \subset \mathcal{M}$$

is a homeomorphism of $\mathcal{U}$ *onto an open set* $\mathbf{x}(\mathcal{U})$ *of* $\mathcal{M}$*.*

Here $\mathbf{x}$ is called the ***local homeomorphism*** of the patch, and $\mathbf{x}(\mathcal{U})$ the ***coordinate neighborhood***. Frequently, we refer to 'the patch $\mathbf{x}$' when the domain $\mathcal{U}$ is understood. Let

$$x_i = u_i \circ \mathbf{x}^{-1} \colon \mathbf{x}(\mathcal{U}) \longrightarrow \mathbb{R} \tag{24.1}$$

for $i = 1, \ldots, n$. Then x_i is called the i^{th} ***coordinate function*** and $(x_1, \ldots, x_n)$ is called a ***system of local coordinates*** for $\mathcal{M}$. The coordinate functions $x_1, \ldots, x_n$ contain the same information as the local homeomorphism $\mathbf{x}$. Often, we write $\mathbf{x}^{-1} = (x_1, \ldots, x_n)$, with the meaning that

$$\mathbf{x}^{-1}(\mathbf{p}) = \big(x_1(\mathbf{p}), \ldots, x_n(\mathbf{p})\big)$$

for all $\mathbf{p} \in \mathbf{x}(\mathcal{U})$.

We are now ready to define the notion of differentiable manifold. Intuitively, the idea is this: in studying the geography of the earth, it is a great convenience to use geographical maps or *patches* instead of examining the earth directly. A collection of geographical maps that covers the earth is called an ***atlas***, and it gives a complete picture of the earth. Roughly, we shall follow the same procedure with differentiable manifolds.

Definition 24.2. *An* ***atlas*** $\mathfrak{A}$ *on a topological space* $\mathcal{M}$ *is a collection of patches on* $\mathcal{M}$ *such that all the patches map from open subsets of the same Euclidean space* $\mathbb{R}^n$ *into* $\mathcal{M}$*, and* $\mathcal{M}$ *is the union of all the* $\mathbf{x}(\mathcal{U})$*'s such that* $(\mathbf{x},\mathcal{U}) \in \mathfrak{A}$*. A topological space* $\mathcal{M}$ *equipped with an atlas is called a* ***topological manifold***.

Let $\mathfrak{A}$ be an atlas on a topological space $\mathcal{M}$. Notice that if $(\mathbf{x},\mathcal{U})$ and $(\mathbf{y},\mathcal{V})$ are two patches in $\mathfrak{A}$ such that $\mathbf{x}(\mathcal{U}) \cap \mathbf{y}(\mathcal{V}) = \mathcal{W}$ is a nonempty subset of $\mathcal{M}$, then the map

$$\mathbf{x}^{-1} \circ \mathbf{y} \colon \mathbf{y}^{-1}(\mathcal{W}) \longrightarrow \mathbf{x}^{-1}(\mathcal{W}) \tag{24.2}$$

is a homeomorphism between open subsets of $\mathbb{R}^n$. We call $\mathbf{x}^{-1} \circ \mathbf{y}$ a ***change of coordinates***. We are now in a position to state

Definition 24.3. *A* ***differentiable manifold*** *is a paracompact topological space* $\mathcal{M}$ *equipped with an atlas* $\mathfrak{A}$ *such that for any two patches* $(\mathbf{x},\mathcal{U}), (\mathbf{y},\mathcal{V})$ *in* $\mathfrak{A}$ *with* $\mathbf{x}(\mathcal{U}) \cap \mathbf{y}(\mathcal{V}) = \mathcal{W}$ *nonempty, the change of coordinates* (24.2) *is differentiable (that is, of class* C^∞*) in the ordinary Euclidean sense. The* ***dimension*** *of the manifold* $\mathcal{M}$ *(denoted by* $\dim \mathcal{M}$*) is the number n in Definition 24.2.*

Remarks

(1) Topological terms applied to a manifold apply to its underlying topological space; for example, we shall assume that a given differentiable manifold is connected, unless stated otherwise. Manifolds share many of the local properties of Euclidean space $\mathbb{R}^n$, for example, local connectedness and

local compactness, but the paracompact assumption needs to be added. Strictly speaking, the hypothesis of Hausdorff should also be added in the definition to exclude certain pathological examples. The exact meaning of these words is not important for us, but their definitions can be found in any book on general topology, such as [Kelley].

(2) A function of n complex variables is called ***holomorphic*** if it is so in each variable separately (see page 721), and in this case it is expandable in a complex power series in a neighborhood of each point where it is defined. The definition of ***complex manifold*** is the same as that of a differentiable manifold, except that the local homeomorphisms are required to map from open subsets of $\mathbb{C}^n$, and the changes of coordinates (24.2) are required to be holomorphic rather than C^∞. Any such complex manifold is, in particular, a real manifold of dimension $2n$.

Implicitly associated with $\mathfrak{A}$ is a possibly larger atlas, which is a theoretical nicety that is needed to declare when two manifolds are really the same.

Definition 24.4. *The* ***completion*** $\widetilde{\mathfrak{A}}$ *of an atlas* $\mathfrak{A}$ *is the collection*

$$\widetilde{\mathfrak{A}} = \left\{ (\mathbf{x},\mathcal{U}) \mid \mathbf{x}^{-1}\circ\mathbf{y} \text{ and } \mathbf{y}^{-1}\circ\mathbf{x} \text{ are differentiable for all } (\mathbf{y},\mathcal{V}) \in \mathfrak{A} \right\}.$$

We say that an atlas $\mathfrak{A}$ *is* ***complete*** *if it coincides with its completion.*

We do not distinguish between differentiable manifolds $(\mathcal{M},\mathfrak{A}_1)$ and $(\mathcal{M},\mathfrak{A}_2)$ when the atlases $\mathfrak{A}_1$ and $\mathfrak{A}_2$ have the same completion. When we speak of a 'differentiable manifold $\mathcal{M}$', we shall mean a differentiable manifold $\mathcal{M}$ equipped with a specific complete atlas.

There are two simple but important ways to construct new manifolds from old.

Definition 24.5. *Let* $\mathcal{M}$ *be a differentiable manifold defined by a complete atlas* $\mathfrak{A}$*, and let* $\mathcal{V}$ *be an open subset of* $\mathcal{M}$*. Define*

$$\mathfrak{A}|\mathcal{V} = \left\{ (\mathbf{x},\mathcal{U}) \in \mathfrak{A} \mid \mathbf{x}(\mathcal{U}) \subseteq \mathcal{V} \right\}.$$

Evidently, $\mathfrak{A}|\mathcal{V}$ *is an atlas of* $\mathcal{V}$*. We call* $\mathcal{V}$ *equipped with the atlas* $\mathfrak{A}(V)$ *an* ***open submanifold*** *of* $\mathcal{M}$*.*

Definition 24.6. *Let* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *be differentiable manifolds of dimensions* n_1 *and* n_2 *defined by atlases* $\mathfrak{A}_1$ *and* $\mathfrak{A}_2$*. If* $(\mathbf{x},\mathcal{U})$ *is in* $\mathfrak{A}_1$ *and* $(\mathbf{y},\mathcal{V})$ *is in* $\mathfrak{A}_2$*, we define* $\mathbf{x}\times\mathbf{y}\colon \mathcal{U}\times\mathcal{V}\to\mathcal{M}_1\times\mathcal{M}_2$ *by*

$$(\mathbf{x}\times\mathbf{y})(\mathbf{p},\mathbf{q}) = \big(\mathbf{x}(\mathbf{p}),\mathbf{y}(\mathbf{q})\big).$$

Then $\mathcal{M}_1 \times \mathcal{M}_2$ equipped with the completion of the atlas

$$\{ (\mathbf{x} \times \mathbf{y}, \mathcal{U} \times \mathcal{V}) \mid (\mathbf{x}, \mathcal{U}) \in \mathfrak{A}_1,\ (\mathbf{y}, \mathcal{V}) \in \mathfrak{A}_2 \}$$

is a differentiable manifold of dimension $n_1 + n_2$, called the ***product*** *of $\mathcal{M}_1$ and $\mathcal{M}_2$.*

For the proof that $\mathcal{M}_1 \times \mathcal{M}_2$ is actually a differentiable manifold, see Exercise 1.

Examples of Differentiable Manifolds

The Euclidean space $\mathbb{R}^n$ is made into a differentiable manifold as follows. Let $\mathbf{1}\colon \mathbb{R}^n \to \mathbb{R}^n$ denote the identity map

$$(u_1, , \dots, u_n)\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n.$$

Then $(\mathbf{1}, \mathbb{R}^n)$ constitutes an atlas for $\mathbb{R}^n$ all by itself. This formalizes the fact that the notion of manifold is a generalization of Euclidean space.

The n-dimensional sphere represents an example where the concept of atlas is essential. Let $a > 0$ and put

$$S^n(a) = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \;\middle|\; \sum_{j=0}^{n} t_j^2 = a^2 \right\}.$$

We shall make $S^n(a)$ into a differentiable manifold by defining an atlas

$$\mathfrak{A} = \{ (\mathsf{north}, \mathbb{R}^n),\ (\mathsf{south}, \mathbb{R}^n) \},$$

where north and south are the 'stereographic injections'. Analytically these are patches

$$\mathsf{north}\colon \mathbb{R}^n \longrightarrow S^n(a) \setminus \{\mathbf{n}\} \qquad \text{and} \qquad \mathsf{south}\colon \mathbb{R}^n \longrightarrow S^n(a) \setminus \{\mathbf{s}\},$$

where $\mathbf{n} = (a, 0, \dots, 0)$ is the 'north pole', and $\mathbf{s} = (-a, 0, , \dots, 0)$ is the 'south pole'. They are defined by setting

$$\mathsf{north} = (\Phi_0, \dots, \Phi_n), \qquad \mathsf{south} = (\Psi_0, \dots, \Psi_n),$$

where

$$\Phi_0 = -\Psi_0 = a \left(\frac{-a^2 + \displaystyle\sum_{j=1}^{n} u_j^2}{a^2 + \displaystyle\sum_{j=1}^{n} u_j^2} \right), \qquad \text{and} \qquad \Phi_k = \Psi_k = \frac{2a^2 u_k}{a^2 + \displaystyle\sum_{j=1}^{n} u_j^2}$$

for $k = 1, \dots, n$. In the special case $n = 2$ and $a = 1$, north coincides with $\boldsymbol{\Upsilon}$ on page 250. It is also the inverse of the stereographic projection described by

Definition 22.19 and Figure 22.2 on page 730 with (u_1, u_2) a point of the equatorial plane $\mathbb{R}^2$; south is obtained using straight lines passing through $(0, 0, -a)$ instead of $(a, 0, 0)$. It is easily verified that north and south are injective, and the compositions $\mathsf{north} \circ \mathsf{south}^{-1}$ and $\mathsf{south} \circ \mathsf{north}^{-1}$ are differentiable.

To conclude this subsection, we insert a result that provides a whole host of examples.

Lemma 24.7. *A regular surface $\mathcal{M}$ in $\mathbb{R}^n$ is a differentiable manifold.*

Proof. A regular surface in $\mathbb{R}^n$ was defined on page 297. For the atlas of $\mathcal{M}$ we choose the regular injective patches on $\mathcal{M}$. Corollary 10.30, page 300, implies that each change of coordinates is differentiable. Hence $\mathcal{M}$ is a differentiable manifold. ∎

24.2 Differentiable Functions on Manifolds

Differentiable manifolds are locally like Euclidean space. For their study it will be important to transfer to manifolds as much of the differential calculus of Euclidean space as we can. First on the agenda is the notion of differentiability for real-valued functions on a differentiable manifold. The definition of this notion is almost the same as the corresponding definition that we gave on page 301.

Definition 24.8. *Let $f\colon \mathcal{W} \to \mathbb{R}$ be a function defined on an open subset $\mathcal{W}$ of a differentiable manifold $\mathcal{M}$. We say that f is **differentiable** at $\mathbf{p} \in \mathcal{W}$, provided that for some patch $\mathbf{x}\colon \mathcal{U} \to \mathcal{M}$ with $\mathcal{U} \subset \mathbb{R}^n$ and $\mathbf{p} \in \mathbf{x}(\mathcal{U}) \subset \mathcal{W}$, the composition $f \circ \mathbf{x}\colon \mathcal{U} \to \mathbb{R}$ is differentiable (in the ordinary Euclidean sense) at $\mathbf{x}^{-1}(\mathbf{p})$. If f is differentiable at all points of $\mathcal{W}$, we say that f is **differentiable** on $\mathcal{W}$.*

This definition illustrates the force behind the definition of differentiable manifold. One might think that it should be necessary to require that $f \circ \mathbf{x}$ be differentiable for *every* patch $(\mathbf{x}, \mathcal{U})$ in the atlas of $\mathcal{M}$. However, this fact is a consequence of the definition:

Lemma 24.9. *The definition of differentiability of a real-valued function on a differentiable manifold does not depend on the choice of patch.*

Proof. If $(\mathbf{x}, \mathcal{U})$ and $(\mathbf{y}, \mathcal{V})$ are patches on a manifold $\mathcal{M}$, then the change of coordinates $\mathbf{x}^{-1} \circ \mathbf{y}$ is differentiable by the definition of differentiable manifold. We can write $f \circ \mathbf{y}$ as

$$f \circ \mathbf{y} = (f \circ \mathbf{x}) \circ (\mathbf{x}^{-1} \circ \mathbf{y}).$$

Since the composition of the Euclidean-differentiable functions is differentiable, the differentiability of $f \circ \mathbf{x}$ implies the differentiability of $f \circ \mathbf{y}$, and conversely. ∎

Next, we consider the totality of differentiable functions on a differentiable manifold and define some algebraic structure on it.

Definition 24.10. *Let $\mathcal{M}$ be a differentiable manifold. We put*

$$\mathfrak{F}(\mathcal{M}) = \{\, f\colon \mathcal{M} \to \mathbb{R} \mid f \text{ is differentiable} \,\}.$$

*We call $\mathfrak{F}(\mathcal{M})$ the **algebra of real-valued differentiable functions** $\mathcal{M}$.*

For $a, b \in \mathbb{R}$ and $f, g \in \mathfrak{F}(\mathcal{M})$ the functions $af + bg$ and fg are defined by

$$(af + bg)(\mathbf{x}) = af(\mathbf{x}) + bg(\mathbf{x}) \qquad \text{and} \qquad (fg)(\mathbf{x}) = f(\mathbf{x})g(\mathbf{x})$$

for $\mathbf{x} \in \mathcal{M}$. Also, we identify any $a \in \mathbb{R}$ with the constant function a given by $a(\mathbf{x}) = a$ for $\mathbf{x} \in \mathcal{M}$.

Let us note some of the algebraic properties of $\mathfrak{F}(\mathcal{M})$.

Lemma 24.11. *Let $\mathcal{M}$ be a differentiable manifold. Then $\mathfrak{F}(\mathcal{M})$ is a commutative ring with identity and an algebra over the real numbers $\mathbb{R}$.*

Proof. Let $f, g \in \mathfrak{F}(\mathcal{M})$ and $a, b \in \mathbb{R}$. If $(\mathbf{x}, \mathcal{U})$ is a patch on $\mathcal{M}$, then $f \circ \mathbf{x}$ and $g \circ \mathbf{x}$ are differentiable in the ordinary Euclidean sense; hence $a(f \circ \mathbf{x}) + b(g \circ \mathbf{x})$ and $(f \circ \mathbf{x})(g \circ \mathbf{x})$ are Euclidean differentiable. It follows easily that both $af + bg$ and fg are differentiable. Also, constant functions are differentiable, and the identity of the ring $\mathfrak{F}(\mathcal{M})$ is $1 \in \mathbb{R}$. Associativity, commutativity and distributivity are easy to prove. ∎

Note also that if $f \in \mathfrak{F}(\mathcal{M})$ is never zero, then $1/f \in \mathfrak{F}(\mathcal{M})$.

It will be important to know when we can extend a real-valued function on an open set of $\mathcal{M}$ to a function that is differentiable on *all* of $\mathcal{M}$.

Lemma 24.12. *Let $\mathcal{M}$ be a differentiable manifold, and let $\mathbf{p} \in \mathcal{M}$. If $\mathcal{W}$ is an open neighborhood of $\mathbf{p}$, then there exist a function $k \in \mathfrak{F}(\mathcal{M})$ and open sets $\mathcal{P}$ and $\mathcal{Q}$ such that $\mathbf{p} \in \mathcal{P} \subset \mathcal{M} \setminus \mathcal{Q} \subset \mathcal{W}$ and*

(i) $0 \leqslant k(\mathbf{x}) \leqslant 1$ *for* $\mathbf{x} \in \mathcal{M}$;

(ii) $k(\mathbf{x}) = 1$ *for* $\mathbf{x} \in \mathcal{P}$;

(iii) $k(\mathbf{x}) = 0$ *for* $\mathbf{x} \in \mathcal{Q}$.

Proof. For $c > 0$ let

$$\mathcal{B}_c = \left\{ (p_1, \ldots, p_n) \in \mathbb{R}^n \;\middle|\; \sum_{j=1}^{n} p_j^2 < c \right\}.$$

There exist real numbers a, b with $0 < a < b$ and patches $(\mathbf{x}, \mathcal{B}_a)$, $(\mathbf{x}, \mathcal{B}_b)$ in the (completion of the) atlas of $\mathcal{M}$ such that

$$\mathbf{p} \in \mathbf{x}(\mathcal{B}_a) \subset \mathbf{x}(\mathcal{B}_b) \subset \overline{\mathbf{x}(\mathcal{B}_b)} \subset \mathcal{W},$$

where $\overline{\mathbf{x}(\mathcal{B}_b)}$ is the closure of $\mathbf{x}(\mathcal{B}_b)$. Set $\mathcal{P} = \mathbf{x}(\mathcal{B}_a)$ and $\mathcal{Q} = \mathcal{M} \setminus \overline{\mathbf{x}(\mathcal{B}_b)}$.

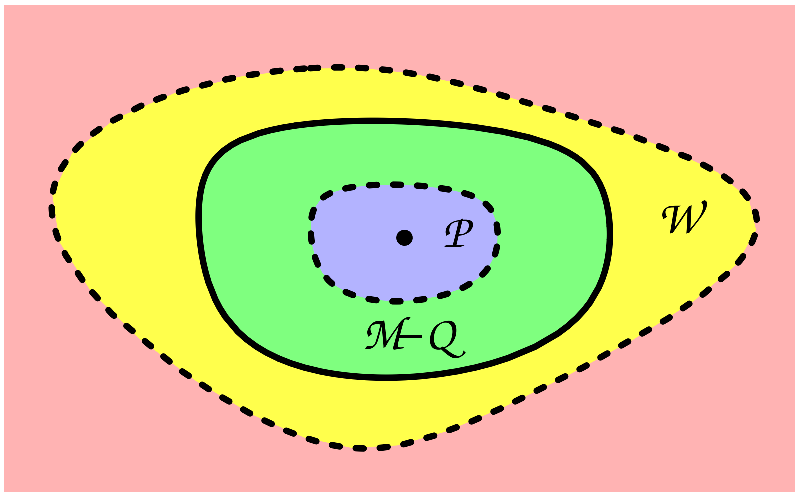


Figure 24.1: Neighborhoods of $\mathbf{p}$ and $\mathcal{M} \setminus \mathcal{W}$

Define $g\colon \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} \exp\left(\dfrac{1}{x-b} - \dfrac{1}{x-a}\right) & \text{if } a < x < b, \\ 0 & \text{otherwise.} \end{cases}$$

Then g is a C^∞ function (but not a C^ω one). This fact is obvious at all points except a and b. It is clear at least that g is continuous at a and b. Moreover, it follows from L'Hôpital's rule that

$$\begin{aligned}
\lim_{x \downarrow a} \frac{g(x) - g(a)}{x - a} &= \lim_{x \downarrow a} \frac{\exp\left(\dfrac{1}{x-b} - \dfrac{1}{x-a}\right)}{x-a} \\
&= \exp\left(\frac{1}{a-b}\right) \lim_{x \downarrow a} \frac{\exp\left(-\dfrac{1}{x-a}\right)}{x-a} = \exp\left(\frac{1}{a-b}\right) \lim_{y \downarrow 0} \frac{\dfrac{1}{y}}{e^{\frac{1}{y}}} \\
&= \exp\left(\frac{1}{a-b}\right) \lim_{y \downarrow 0} \frac{\dfrac{-1}{y^2}}{\dfrac{-1}{y^2} e^{\frac{1}{y}}} = \exp\left(\frac{1}{a-b}\right) \lim_{y \downarrow 0} e^{-\frac{1}{y}} = 0.
\end{aligned}$$

In a similar fashion, it can be shown that all other derivatives of g exist and vanish at both a and b. The functions $G\colon \mathbb{R} \to \mathbb{R}$ and $\psi\colon \mathbb{R}^n \to \mathbb{R}$ defined by

$$G(x) = \frac{\displaystyle\int_x^b g(t)\,dt}{\displaystyle\int_a^b g(t)\,dt}, \qquad \psi(p_1, \ldots, p_n) = G\big(p_1^2 + \cdots + p_n^2\big)$$

are also differentiable. We define $k\colon \mathcal{M} \to \mathbb{R}$ by

$$k(\mathbf{q}) = \begin{cases} \psi\big(\mathbf{x}^{-1}(\mathbf{q})\big), & \text{for } \mathbf{q} \in \mathbf{x}(\mathcal{B}_b), \\ 0, & \text{otherwise.} \end{cases}$$

Then k has all the required properties. ▮

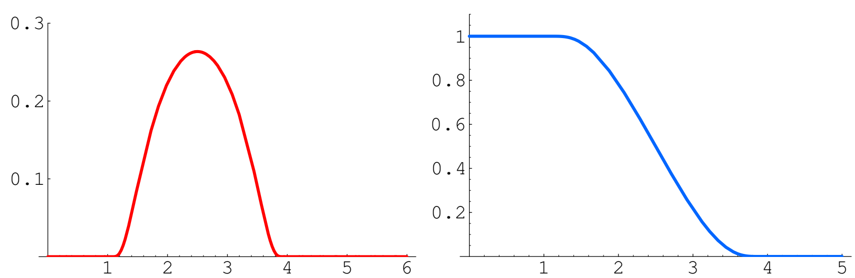


Figure 24.2: The functions $g(x)$ and $G(x)$

If f is a real-valued differentiable function on an open subset $\mathcal{W}$ of a differentiable manifold $\mathcal{M}$, it might not be possible to extend f to a differentiable function defined on all of $\mathcal{M}$, as there might be some bizarre behavior of f at the boundary of $\mathcal{W}$. However, we now show that each point of $\mathcal{W}$ has a neighborhood $\mathcal{P}$ such that we can extend the *restriction* $f|\mathcal{P}$ to all of $\mathcal{M}$.

Lemma 24.13. Let $\mathcal{W} \subset \mathcal{M}$ be an open neighborhood of $\mathbf{p} \in \mathcal{M}$, and suppose that $f \in \mathfrak{F}(\mathcal{W})$. Then there exist $\widetilde{f} \in \mathfrak{F}(\mathcal{M})$ and an open set $\mathcal{P}$ with $\mathbf{p} \in \mathcal{P} \subseteq \mathcal{W}$ such that $\widetilde{f}|\mathcal{P} = f|\mathcal{P}$. We call $\widetilde{f}$ a ***globalization*** *of f.*

Proof. By Lemma 24.12 there are neighborhoods $\mathcal{P}$ of $\mathbf{p}$ and $\mathcal{Q}$ of $\mathcal{M} \setminus \mathcal{W}$ with the properties:

(i) $\mathcal{P} \subseteq \mathcal{W}$;

(ii) there is a function $k \in \mathfrak{F}(\mathcal{M})$ such that $k(\mathbf{x}) = 1$ for $\mathbf{x} \in \mathcal{P}$ and $k(\mathbf{x}) = 0$ for $\mathbf{x} \in \mathcal{Q}$.

Define $\widetilde{f}\colon \mathcal{M} \to \mathbb{R}$ by

$$\widetilde{f}(\mathbf{q}) = \begin{cases} (kf)(\mathbf{q}) & \text{for } \mathbf{q} \in \mathcal{W}, \\ 0 & \text{for } \mathbf{q} \in \mathcal{M} \setminus \mathcal{W}. \end{cases}$$

Then $\widetilde{f} \in \mathfrak{F}(\mathcal{M})$, and on $\mathcal{P}$ we have $\widetilde{f} = kf = f$. ▮

The coordinate functions (24.1) are elements of $\mathfrak{F}(\mathbf{x}(\mathcal{U}))$. By applying Lemma 24.13 to the monomials $(x_i)^m$, we see that $\mathfrak{F}(\mathcal{M})$ is *infinite-dimensional*. By contrast, the analogs of Lemmas 24.12 and 24.13 are false for C^ω functions.

Having defined the notion of real-valued differentiable function on a differentiable manifold, we are ready to define what it means for a map between manifolds to be differentiable.

Definition 24.14. *Let $\mathcal{M}$,$\mathcal{N}$ be differentiable manifolds, and let $\Psi\colon \mathcal{M} \to \mathcal{N}$ be a map. We say that Ψ is* ***differentiable*** *provided $\mathbf{y}^{-1} \circ \Psi \circ \mathbf{x}$ is differentiable for every patch $(\mathbf{x},\mathcal{U})$ in the atlas of $\mathcal{M}$ and every patch $(\mathbf{y},\mathcal{V})$ in the atlas of $\mathcal{N}$, where the compositions are defined. A* ***diffeomorphism*** *between manifolds $\mathcal{M}$ and $\mathcal{N}$ is a differentiable map $\Phi\colon \mathcal{M} \to \mathcal{N}$ which has a differentiable inverse $\Phi^{-1}\colon \mathcal{N} \to \mathcal{M}$. If such a map Φ exists, $\mathcal{M}$ and $\mathcal{N}$ are said to be* ***diffeomorphic****. A map $\Psi\colon \mathcal{M} \to \mathcal{N}$ is called a* ***local diffeomorphism*** *provided each $\mathbf{p} \in \mathcal{M}$ has a neighborhood $\mathcal{W}$ such that $\Psi|\mathcal{W}\colon \mathcal{W} \to \Psi(\mathcal{W})$ is a diffeomorphism.*

The following lemma is an easy consequence of the definitions and the fact that the corresponding lemma for $\mathbb{R}^n$ is known.

Lemma 24.15. *Suppose $\mathcal{M} \xrightarrow{\Phi} \mathcal{N} \xrightarrow{\Psi} \mathcal{P}$ are differentiable maps between differentiable manifolds. Then the composition $\Psi \circ \Phi\colon \mathcal{M} \to \mathcal{P}$ is differentiable. If Φ and Ψ are diffeomorphisms, then so is $\Psi \circ \Phi$ and*

$$(\Psi \circ \Phi)^{-1} = \Phi^{-1} \circ \Psi^{-1}.$$

The coordinates (24.1) are examples of differentiable functions $\mathbf{x}(\mathcal{U}) \to \mathbb{R}$. Moreover,

Lemma 24.16. *Let $\mathcal{M}$ be an n-dimensional differentiable manifold, and let $\mathbf{x}\colon \mathcal{U} \to \mathcal{M}$ be a patch. Write $\mathbf{x}^{-1} = (x_1, \ldots, x_n)$. Then*

(i) *$\mathbf{x}$ is a differentiable mapping between the manifolds $\mathcal{U}$ and $\mathcal{M}$;*

(ii) *$\mathbf{x}^{-1}\colon \mathbf{x}(\mathcal{U}) \to \mathbb{R}^n$ is differentiable;*

Proof. Let $\mathbf{1}\colon \mathbb{R}^n \to \mathbb{R}^n$ be the identity map. For each patch $(\mathbf{y}, \mathcal{V})$ in $\mathfrak{A}$, the maps $\mathbf{y}^{-1} \circ \mathbf{x} \circ \mathbf{1}$ and $\mathbf{1} \circ \mathbf{x}^{-1} \circ \mathbf{y}$ are $\mathbb{R}^n$-differentiable. By definition $\mathbf{x}$ and $\mathbf{x}^{-1}$, considered as maps between the manifolds $\mathcal{U}$ and $\mathcal{M}$, are differentiable. ▮

A differentiable map between manifolds induces a correspondence between the algebras of differentiable functions on each manifold.

Lemma 24.17. *Let $\Phi\colon \mathcal{M} \to \mathcal{N}$ be a differentiable mapping between manifolds. Then $f \in \mathfrak{F}(\mathcal{N})$ implies $f \circ \Phi \in \mathfrak{F}(\mathcal{M})$.*

Proof. Let $(\mathbf{x}, \mathcal{U})$ be a patch on $\mathcal{M}$ and $(\mathbf{y}, \mathcal{V})$ a patch on $\mathcal{N}$. By hypothesis, $f \circ \mathbf{y}\colon \mathcal{V} \to \mathbb{R}$ and $\mathbf{y}^{-1} \circ \Phi \circ \mathbf{x}$ is differentiable. Hence the composition

$$f \circ \Phi \circ \mathbf{x} = (f \circ \mathbf{y}) \circ (\mathbf{y}^{-1} \circ \Phi \circ \mathbf{x})$$

is differentiable. Since this is true for every patch $(\mathbf{x}, \mathcal{U})$ in the atlas of $\mathcal{M}$, it follows that $f \circ \Phi$ is differentiable. ▮

24.3 Tangent Vectors on Manifolds

By their very nature differentiable manifolds are 'curved' spaces, and they can be very complicated objects to study. In comparison, a vector space is much simpler. But a vector space has a great deal of structure that facilitates the study of differentiable manifolds. In this section, we define the notion of tangent space to a differentiable manifold $\mathcal{M}$ at a point $\mathbf{p} \in \mathcal{M}$. It can be thought of as the best linear approximation to $\mathcal{M}$ at $\mathbf{p}$.

The notion of a vector tangent to a curve or surface in $\mathbb{R}^n$ is intuitively clear, as we have seen in Chapters 1, 7 and 10; we want to define a similar concept for an arbitrary differentiable manifold. However, if we try to generalize directly the notion of tangent vector, we face a difficulty: the elementary definition of tangent vector that we gave in Section 10.5 makes a tangent vector to a surface a tangent vector to $\mathbb{R}^n$. But a surface, or more generally an arbitrary manifold, is not *a priori* contained in any Euclidean space, so we need a definition of tangent vector that does not depend on any such assumption.

Working backwards, let us suppose we have a suitable definition of a vector $\mathbf{v}_{\mathbf{p}}$ tangent to a manifold $\mathcal{M}$ at a point $\mathbf{p} \in \mathcal{M}$. Then, just as in elementary calculus, we can speak of the derivative $\mathbf{v}_{\mathbf{p}}[f]$ of $f \in \mathfrak{F}(\mathcal{M})$ in the direction $\mathbf{v}_{\mathbf{p}}$. Roughly speaking, the number $\mathbf{v}_{\mathbf{p}}[f]$ is the ordinary derivative of f at $\mathbf{p}$ along a curve leaving $\mathbf{p}$ in the direction $\mathbf{v}_{\mathbf{p}}$. Now for a fixed $\mathbf{v}_{\mathbf{p}}$, the function $\mathfrak{F}(\mathcal{M}) \to \mathbb{R}$ that maps f into $\mathbf{v}_{\mathbf{p}}[f]$ has the essential properties always possessed by differentiation: linearity and the Leibniz[2] product rule. Thus, in standard mathematical fashion, we shall define a tangent vector to be a function that has

[2] 

Baron Gottfried Wilhelm Leibniz (1646–1716). German mathematician. A cofounder of calculus. Although Leibniz discovered calculus a few years later than Newton, it is the notation of Leibniz (such as dt and $\int$) that has gained the widest acceptance.

these properties. Although it is not clear from the outset that this definition will yield an object that has all the intuitive properties of a tangent vector, in fact, it does.

Definition 24.18. *Let* $\mathbf{p}$ *be a point of a manifold* $\mathcal{M}$. *A* ***tangent vector*** $\mathbf{v_p}$ *to* $\mathcal{M}$ *at* $\mathbf{p}$ *is a real-valued function* $\mathbf{v_p}\colon \mathfrak{F}(\mathcal{M}) \to \mathbb{R}$ *such that*

$$\begin{aligned} \mathbf{v_p}[af + bg] &= a\,\mathbf{v_p}[f] + b\,\mathbf{v_p}[g] && \text{(the \textbf{\textit{linearity}} property)},\\ \mathbf{v_p}[fg] &= f(\mathbf{p})\mathbf{v_p}[g] + g(\mathbf{p})\mathbf{v_p}[f] && \text{(the \textbf{\textit{Leibnizian}} property)}, \end{aligned}$$

for all $a, b \in \mathbb{R}$ *and* $f, g \in \mathfrak{F}(\mathcal{M})$.

Clearly, a tangent vector to a surface in $\mathbb{R}^3$ as defined in Section 10.5 gives rise to a tangent vector in the sense of this definition.

We can easily give some nontrivial examples of tangent vectors. First, we need some new notation.

Definition 24.19. *Let* $\mathbf{x}\colon \mathcal{U} \to \mathcal{M}$ *be a patch on a differentiable manifold* $\mathcal{M}$ *and write* $\mathbf{x}^{-1} = (x_1, \ldots, x_n)$. *For* $f \in \mathfrak{F}(\mathcal{M})$ *and* $\mathbf{p} \in \mathbf{x}(\mathcal{U})$, *write* $\mathbf{p} = \mathbf{x}(\mathbf{q})$, *and define*

$$\frac{\partial f}{\partial x_i}(\mathbf{p}) = \left.\frac{\partial f}{\partial x_i}\right|_{\mathbf{p}} = \left.\frac{\partial (f \circ \mathbf{x})}{\partial u_i}\right|_{\mathbf{q}} \tag{24.3}$$

for $i = 1, \ldots, n$. *Here, as usual, the* u_i*'s are the natural coordinate functions of* $\mathbb{R}^n$, *and the ordinary Euclidean partial derivative appears on the right-hand side of* (24.3).

Note that for $\mathcal{M} = \mathbb{R}^n$ and $\mathbf{x}$ the identity map, the right-hand side of (24.3) reduces to the ordinary partial derivative. In general, we can write

$$\frac{\partial f}{\partial x_i} = \frac{\partial (f \circ \mathbf{x})}{\partial u_i} \circ \mathbf{x}^{-1}.$$

Lemma 24.20. *Let* $\mathbf{x}\colon \mathcal{U} \to \mathcal{M}$ *be a patch on a differentiable manifold* $\mathcal{M}$ *and write* $\mathbf{x}^{-1} = (x_1, \ldots, x_n)$. *For* $\mathbf{p} \in \mathbf{x}(\mathcal{U})$, *and for each* $i = 1, \ldots, n$, *the function*

$$\left.\frac{\partial}{\partial x_i}\right|_{\mathbf{p}} : \mathfrak{F}(\mathcal{M}) \longrightarrow \mathbb{R}$$

defined by

$$\left.\frac{\partial}{\partial x_i}\right|_{\mathbf{p}}[f] = \left.\frac{\partial f}{\partial x_i}\right|_{\mathbf{p}}$$

is a tangent vector to $\mathcal{M}$ *at* $\mathbf{p}$.

Proof. Put $\mathbf{p}=\mathbf{x}(\mathbf{q})$, where $\mathbf{q}\in\mathcal{U}$. Let $a,b\in\mathbb{R}$ and $f,g\in\mathfrak{F}(\mathcal{M})$. We have

$$\begin{aligned}\left.\frac{\partial(af+bg)}{\partial x_i}\right|_{\mathbf{p}} &= \left.\frac{\partial}{\partial u_i}\big((af+bg)\circ\mathbf{x}\big)\right|_{\mathbf{q}} = \left.\frac{\partial}{\partial u_i}\big(a(f\circ\mathbf{x})+b(g\circ\mathbf{x})\big)\right|_{\mathbf{q}}\\ &= a\left.\frac{\partial(f\circ\mathbf{x})}{\partial u_i}\right|_{\mathbf{q}}+b\left.\frac{\partial(g\circ\mathbf{x})}{\partial u_i}\right|_{\mathbf{q}} = a\left.\frac{\partial f}{\partial x_i}\right|_{\mathbf{p}}+b\left.\frac{\partial g}{\partial x_i}\right|_{\mathbf{p}}.\end{aligned}$$

Similarly,

$$\begin{aligned}\left.\frac{\partial(fg)}{\partial x_i}\right|_{\mathbf{p}} &= \left.\frac{\partial}{\partial u_i}\big((fg)\circ\mathbf{x}\big)\right|_{\mathbf{q}} = \left.\frac{\partial}{\partial u_i}\big((f\circ\mathbf{x})(g\circ\mathbf{x})\big)\right|_{\mathbf{q}}\\ &= (f\circ\mathbf{x})(\mathbf{q})\left.\frac{\partial(g\circ\mathbf{x})}{\partial u_i}\right|_{\mathbf{q}}+(g\circ\mathbf{x})(\mathbf{q})\left.\frac{\partial(f\circ\mathbf{x})}{\partial u_i}\right|_{\mathbf{q}}\\ &= f(\mathbf{p})\left.\frac{\partial g}{\partial x_i}\right|_{\mathbf{p}}+g(\mathbf{p})\left.\frac{\partial f}{\partial x_i}\right|_{\mathbf{p}}.\ \blacksquare\end{aligned}$$

Now that we know that there exist some tangent vectors at every point $\mathbf{p}$, let us consider the set of *all* tangent vectors at $\mathbf{p}$.

*Definition 24.21. Let $\mathcal{M}$ be a differentiable manifold, and let $\mathbf{p}$ be a point in $\mathcal{M}$. The **tangent space** to $\mathcal{M}$ at $\mathbf{p}$ is the set of all tangent vectors to $\mathcal{M}$ at $\mathbf{p}$. Thus, $\mathcal{M}_{\mathbf{p}}$ equals the set of $\mathbf{v}_{\mathbf{p}}\colon\mathfrak{F}(\mathcal{M})\to\mathbb{R}$ satisfying*

$$\mathbf{v}_{\mathbf{p}}[af+bg]=a\,\mathbf{v}_{\mathbf{p}}[f]+b\,\mathbf{v}_{\mathbf{p}}[g]\quad\textit{and}\quad\mathbf{v}_{\mathbf{p}}[fg]=f(\mathbf{p})\mathbf{v}_{\mathbf{p}}[g]+g(\mathbf{p})\mathbf{v}_{\mathbf{p}}[f],$$

for $a,b\in\mathbb{R}$ and $f,g\in\mathfrak{F}(\mathcal{M})$.

Lemma 24.22. If $\mathcal{M}$ is a differentiable manifold and $\mathbf{p}$ is a point in $\mathcal{M}$, then the tangent space $\mathcal{M}_{\mathbf{p}}$ is naturally a vector space.

Proof. We make $\mathcal{M}_{\mathbf{p}}$ into a vector space over $\mathbb{R}$ as follows. Let $\mathbf{v}_{\mathbf{p}},\mathbf{w}_{\mathbf{p}}\in\mathcal{M}_{\mathbf{p}}$, $a\in\mathbb{R}$ and $f\in\mathfrak{F}(\mathcal{M})$. Then we define $\mathbf{v}_{\mathbf{p}}+\mathbf{w}_{\mathbf{p}}$ and $a\,\mathbf{v}_{\mathbf{p}}$ by

$$\begin{cases}(\mathbf{v}_{\mathbf{p}}+\mathbf{w}_{\mathbf{p}})[f] = \mathbf{v}_{\mathbf{p}}[f]+\mathbf{w}_{\mathbf{p}}[f],\\ (a\,\mathbf{v}_{\mathbf{p}})[f] = a\,\mathbf{v}_{\mathbf{p}}[f].\end{cases}$$

It is easily verified by the same sort of proof as that of Lemma 24.20 that both $\mathbf{v}_{\mathbf{p}}+\mathbf{w}_{\mathbf{p}}$ and $a\,\mathbf{v}_{\mathbf{p}}$ belong to $\mathcal{M}_{\mathbf{p}}$. Thus, addition and scalar multiplication makes $\mathcal{M}_{\mathbf{p}}$ into a vector space. $\blacksquare$

We would now like to prove that $\mathcal{M}_{\mathbf{p}}$ is a finite-dimensional vector space whose dimension is the same as that of $\mathcal{M}$. Furthermore, we have

$$\left\{\left.\frac{\partial}{\partial x_1}\right|_{\mathbf{p}},\ \ldots,\ \left.\frac{\partial}{\partial x_n}\right|_{\mathbf{p}}\right\}$$

as a natural candidate for a basis of $\mathcal{M}_{\mathbf{p}}$. However, there are some technical difficulties to overcome. First, we need

Lemma 24.23. *Let $\mathcal{M}$ be a differentiable manifold and $\mathbf{p} \in \mathcal{M}$.*

(i) *If $f \in \mathfrak{F}(\mathcal{M})$ can be expressed as the product of two functions $g, h \in \mathfrak{F}(\mathcal{M})$ both of which vanish at $\mathbf{p}$, then $\mathbf{v_p}[f] = 0$ for all $\mathbf{v_p} \in \mathcal{M}_{\mathbf{p}}$.*

(ii) *If c is a constant function on $\mathcal{M}$, then $\mathbf{v_p}[c] = 0$ for all $\mathbf{v_p} \in \mathcal{M}_{\mathbf{p}}$.*

(iii) *Tangent vectors are* ***local***; *that is, if $\mathbf{v_p} \in \mathcal{M}_{\mathbf{p}}$, then the number $\mathbf{v_p}[f]$ for $f \in \mathfrak{F}(\mathcal{M})$ depends only on the values of f in a neighborhood of $\mathbf{p}$.*

(iv) *Let h be a differentiable function defined only on a neighborhood $\mathcal{P}$ of $\mathbf{p}$. If $\mathbf{v_p} \in \mathcal{M}_{\mathbf{p}}$, then for any globalization $\widetilde{h}$ of h the value $\mathbf{v_p}[\widetilde{h}]$ is the same; we call this value $\mathbf{v_p}[h]$.*

Proof. For (i) suppose that $f = gh$ where $g, h \in \mathfrak{F}(\mathcal{M})$ and $g(\mathbf{p}) = h(\mathbf{p}) = 0$. Then for $\mathbf{v_p} \in \mathcal{M}_{\mathbf{p}}$ we have by the Leibnizian property that

$$\mathbf{v_p}[f] = \mathbf{v_p}[gh] = g(\mathbf{p})\mathbf{v_p}[h] + h(\mathbf{p})\mathbf{v_p}[g] = 0.$$

Next, for (ii) we first observe that

$$\mathbf{v_p}[1] = \mathbf{v_p}[1^2] = 1 \cdot \mathbf{v_p}[1] + 1 \cdot \mathbf{v_p}[1] = 2\,\mathbf{v_p}[1],$$

so that $\mathbf{v_p}[1] = 0$. For an arbitrary constant c we then have by the linearity property that

$$\mathbf{v_p}[c] = \mathbf{v_p}[c \cdot 1] = c\,\mathbf{v_p}[1] = 0.$$

Since parts (iii) and (iv) are rephrasings of the same statement, we prove (iv). Let $\widetilde{h}$ and $\widehat{h}$ be globalizations of h which agree with h on a neighborhood $\mathcal{W}$ of $\mathbf{p}$, and let $g = \widetilde{h} - \widehat{h}$. Then g vanishes on $\mathcal{W}$. By Lemma 24.12, there exist a neighborhood $\mathcal{P} \subset \mathcal{W}$ of $\mathbf{p}$ and a function $k \in \mathfrak{F}(\mathcal{M})$ such that $k(\mathbf{x}) = 0$ for $\mathbf{x} \in \mathcal{P}$ and $k(\mathbf{x}) = 1$ for $\mathbf{x} \in \mathcal{M} \setminus \mathcal{W}$ (we have replaced k on page 815 by $1 - k$). Then $g = gk$ and $g(\mathbf{p}) = k(\mathbf{p}) = 0$. Hence by part (i) we have $0 = \mathbf{v_p}[g] = \mathbf{v_p}[\widetilde{h}] - \mathbf{v_p}[\widehat{h}]$, so that $\mathbf{v_p}[\widetilde{h}] = \mathbf{v_p}[\widehat{h}]$. ▮

Next, we need a fact about $\mathfrak{F}(\mathbb{R}^n)$.

Lemma 24.24. *Let $g\colon \mathbb{R}^n \to \mathbb{R}$ be a differentiable function, and suppose that $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$. Then there exist differentiable functions $g_i\colon \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, n$ such that*

$$g = \sum_{i=1}^{n} (u_i - a_i) g_i + g(\mathbf{a}). \tag{24.4}$$

Proof. Fix $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n$, and consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(s) = g\bigl(st_1 + (1-s)a_1,\ \ldots\ , st_n + (1-s)a_n\bigr) = g\bigl(s\mathbf{t} + (1-s)\mathbf{a}\bigr).$$

Since g is differentiable, so is f, and

$$f'(s) = \sum_{i=1}^{n}(t_i - a_i)\frac{\partial g}{\partial u_i}\bigl(s\mathbf{t} + (1-s)\mathbf{a}\bigr).$$

Now

$$\int_0^1 f'(s)\,ds = f(1) - f(0) = g(\mathbf{t}) - g(\mathbf{a}),$$

and so the fundamental theorem of calculus implies that

$$\begin{aligned} g(\mathbf{t}) - g(\mathbf{a}) &= \int_0^1 f'(s)\,ds \\ &= \sum_{i=1}^{n}(t_i - a_i)\int_0^1 \frac{\partial g}{\partial u_i}\bigl(s\mathbf{t} + (1-s)\mathbf{a}\bigr)\,ds \\ &= \sum_{i=1}^{n}\bigl(u_i(\mathbf{t}) - a_i\bigr)g_i(\mathbf{t}), \end{aligned}$$

where

$$g_i(\mathbf{t}) = \int_0^1 \frac{\partial g}{\partial u_i}\bigl(s\mathbf{t} + (1-s)\mathbf{a}\bigr)\,ds.$$

Clearly, the g_i's are differentiable. Thus we get (24.4). ∎

Note that the proof of this lemma fails for C^k functions. If g is a C^k function, we can only say that each g_i is a C^{k-1} function.

We are now able to prove:

Theorem 24.25. (Basis Theorem) *Let* $\mathbf{x}\colon \mathcal{U} \to \mathcal{M}$ *be a patch on a differentiable manifold* $\mathcal{M}$ *with* $\mathbf{p} \in \mathbf{x}(\mathcal{U})$ *and write* $\mathbf{x}^{-1} = (x_1, \ldots, x_n)$. *If* $\mathbf{v_p} \in \mathcal{M}_\mathbf{p}$, *then*

$$\mathbf{v_p} = \sum_{i=1}^{n} \mathbf{v_p}[x_i]\frac{\partial}{\partial x_i}\bigg|_\mathbf{p}.$$

Proof. It suffices to show that

$$\mathbf{v_p}[f] = \sum_{i=1}^{n} \mathbf{v_p}[x_i]\frac{\partial f}{\partial x_i}\bigg|_\mathbf{p} \tag{24.5}$$

for all $f \in \mathfrak{F}(\mathcal{M})$. To this end, let $g\colon \mathbb{R}^n \to \mathbb{R}$ be a globalization of $f \circ \mathbf{x}$ near $\mathbf{x}^{-1}(\mathbf{p})$. By Lemma 24.24, there exist differentiable functions $g_i\colon \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, n$ such that

$$g = \sum_{i=1}^{n} \big(u_i - u_i(\mathbf{x}^{-1}(\mathbf{p}))\big) g_i + g\big(\mathbf{x}^{-1}(\mathbf{p})\big).$$

Then near $\mathbf{p}$ we have

$$f = \sum_{i=1}^{n} \big(x_i - x_i(\mathbf{p})\big) f_i + f(\mathbf{p}), \tag{24.6}$$

where $f_i = g_i \circ \mathbf{x}^{-1}$. Lemma 24.23 (iii) allows us to use (24.6) to compute $\mathbf{v_p}[f]$. It follows that

$$\begin{aligned}\mathbf{v_p}[f] &= \sum_{i=1}^{n} \Big\{ f_i(\mathbf{p}) \mathbf{v_p}[x_i - x_i(\mathbf{p})] + \big(x_i(\mathbf{p}) - x_i(\mathbf{p})\big) \mathbf{v_p}[f_i] \Big\} + \mathbf{v_p}[f(\mathbf{p})] \\ &= \sum_{i=1}^{n} f_i(\mathbf{p}) \mathbf{v_p}[x_i],\end{aligned}$$

because $x_i(\mathbf{p})$ and $f(\mathbf{p})$ are constants. For the same reasons, from (24.6) it follows that

$$\left.\frac{\partial f}{\partial x_j}\right|_{\mathbf{p}} = \sum_{i=1}^{n} f_i(\mathbf{p}) \frac{\partial x_i}{\partial x_j}(\mathbf{p}) = f_j(\mathbf{p}),$$

because[3]

$$\frac{\partial x_i}{\partial x_j}(\mathbf{p}) = \frac{\partial (x_i \circ \mathbf{x})}{\partial u_j}\big(\mathbf{x}^{-1}(\mathbf{p})\big) = \frac{\partial u_i}{\partial u_j}\big(\mathbf{x}^{-1}(\mathbf{p})\big) = \delta_{ij}.$$

Hence

$$\left.\left(\sum_{j=1}^{n} \mathbf{v_p}[x_j] \frac{\partial}{\partial x_j} \right)\right|_{\mathbf{p}} [f] = \sum_{j=1}^{n} \mathbf{v_p}[x_j] \left.\frac{\partial f}{\partial x_j}\right|_{\mathbf{p}} = \sum_{j=1}^{n} \mathbf{v_p}[x_j] f_j(\mathbf{p}) = \mathbf{v_p}[f].$$

Thus we get (24.5). ∎

We have seen in Lemmas 24.12 and 24.13 that to show the existence of nonconstant differentiable functions on a differentiable manifold $\mathcal{M}$, it was necessary to use C^∞ functions and not C^ω functions. Furthermore, in the proofs of Lemma 24.24 and Theorem 24.25 it is necessary to use C^∞ functions and not C^k functions.

[3]Here δ_{ij} denotes the Kronecker delta function, defined by $\delta_{ij} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$

Corollary 24.26. *Let* $\mathbf{x}\colon \mathcal{U} \to \mathcal{M}$ *be a patch on a differentiable manifold* $\mathcal{M}$*. Let* $\mathbf{p} \in \mathbf{x}(\mathcal{U})$ *and write* $\mathbf{x}^{-1} = (x_1, \ldots, x_n)$*. Then the vectors*

$$\frac{\partial}{\partial x_1}\bigg|_{\mathbf{p}}, \ldots, \frac{\partial}{\partial x_n}\bigg|_{\mathbf{p}}$$

form a basis for the tangent space $\mathcal{M}_{\mathbf{p}}$*. Hence the dimension of each tangent space* $\mathcal{M}_{\mathbf{p}}$ *as a vector space is the same as the dimension of* $\mathcal{M}$ *as a manifold.*

Proof. The given vectors span the tangent space $\mathcal{M}_{\mathbf{p}}$ by Theorem 24.25. To prove they are linearly independent, suppose that $a_1, \ldots, a_n \in \mathbb{R}$ are such that

$$\sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}\bigg|_{\mathbf{p}} = 0.$$

Then

$$0 = \sum_{i=1}^{n} a_i \frac{\partial x_j}{\partial x_i}\bigg|_{\mathbf{p}} = \sum_{i=1}^{n} a_i \frac{\partial (x_j \circ \mathbf{x})}{\partial u_i}\big(\mathbf{x}^{-1}(\mathbf{p})\big) = \sum_{i=1}^{n} a_i \delta_{ij} = a_j$$

for $j = 1, \ldots, n$. ▮

In order to differentiate functions on manifolds (that is, apply tangent vectors to them) as easily as we would differentiate functions on $\mathbb{R}^n$, we shall need the following result that uses the same notation as Lemma 9.6 on page 267.

Lemma 24.27. (The Chain Rule) *Let* $\mathcal{M}$ *be a differentiable manifold. Suppose* $g_1, \ldots, g_k \in \mathfrak{F}(\mathcal{M})$ *and* $h \in \mathfrak{F}(\mathbb{R}^k)$*. Let* $f = h \circ (g_1, \ldots, g_k)$*. Then for* $\mathbf{v_p} \in \mathcal{M}_{\mathbf{p}}$ *we have*

$$\mathbf{v_p}[f] = \sum_{i=1}^{k} \frac{\partial h}{\partial u_i}\big(g_1(\mathbf{p}), \ldots, g_k(\mathbf{p})\big)\mathbf{v_p}[g_i]. \tag{24.7}$$

Proof. Let $g = (g_1, \ldots, g_k)$. Define $\mathbf{w}\colon \mathfrak{F}(\mathbb{R}^k) \to \mathbb{R}$ by

$$\mathbf{w}[h] = \mathbf{v_p}[h \circ g]$$

for $h \in \mathfrak{F}(\mathbb{R}^k)$. Then it is easy to check from the definition that $\mathbf{w}$ is an element of the tangent space $\mathbb{R}^k_{g(\mathbf{p})}$, because it is linear and Leibnizian. Using Theorem 24.25 with the patch $(u_1, \ldots, u_k)$ at $g(\mathbf{p})$ we get

$$\mathbf{w}[h] = \sum_{i=1}^{k} \mathbf{w}[u_i] \frac{\partial h}{\partial u_i}\big(g(\mathbf{p})\big). \tag{24.8}$$

Now $\mathbf{w}[h] = \mathbf{v_p}[h \circ g] = \mathbf{v_p}[f]$ and $\mathbf{w}[u_i] = \mathbf{v_p}[u_i \circ g] = \mathbf{v_p}[g_i]$, so that (24.8) becomes

$$\mathbf{v_p}[f] = \sum_{i=1}^{k} \frac{\partial h}{\partial u_i}\big(g(\mathbf{p})\big)\mathbf{v_p}[g_i]. \tag{24.9}$$

But (24.9) is another way of writing (24.7). ▮

Note that (24.7) allows us to transfer standard differentiation formulas from $\mathbb{R}^n$ to manifolds. For example, if $\mathbf{v_p}$ is a tangent vector to a manifold $\mathcal{M}$ at $\mathbf{p}$ and $f \in \mathfrak{F}(\mathcal{M})$, then $\mathbf{v_p}[\sin f] = \cos(f(\mathbf{p}))\mathbf{v_p}[f]$.

For abstract surface theory, it will be useful to have alternative notation for tangent vectors.

Definition 24.28. *Let* $\mathbf{x}\colon \mathcal{U} \to \mathcal{M}$ *be a patch on a differentiable manifold* $\mathcal{M}$*, and let* $\mathbf{q} \in \mathcal{U}$*. Then*

$$\mathbf{x}_{u_i}(\mathbf{q})\colon\ \mathfrak{F}(\mathcal{M}) \longrightarrow \mathbb{R}$$

is the operator given by

$$\mathbf{x}_{u_i}(\mathbf{q})[f] = \left.\frac{\partial(f\circ\mathbf{x})}{\partial u_i}\right|_{\mathbf{q}} \tag{24.10}$$

for $\mathbf{q} \in \mathcal{U}$ *and* $f \in \mathfrak{F}(\mathcal{M})$*.*

The point is that (24.10) succeeds in generalizing the derivatives $\mathbf{x}_u, \mathbf{x}_v$ defined by (10.3) on page 289 to the more abstract setting of this chapter.

24.4 Induced Maps

In the previous section, we showed that to each point $\mathbf{p}$ of a differentiable manifold $\mathcal{M}$ there is associated a vector space called the tangent space $\mathcal{M}_{\mathbf{p}}$. In the present section we shall show how a differentiable map $\Psi\colon \mathcal{M} \to \mathcal{N}$ between differentiable manifolds $\mathcal{M}$ and $\mathcal{N}$ gives rise to a linear map between tangent spaces, in complete analogy to Definition 9.9 on page 268. Just as the tangent space is the best linear approximation of a differentiable manifold, the tangent map is the best linear approximation to a differentiable map between manifolds.

Definition 24.29. *Let* $\Psi\colon \mathcal{M} \to \mathcal{N}$ *be a differentiable map between differentiable manifolds* $\mathcal{M}$ *and* $\mathcal{N}$*, and let* $\mathbf{p} \in \mathcal{M}$*. The* ***tangent map*** *of* Ψ *at* $\mathbf{p}$ *is the map*

$$\Psi_{*\mathbf{p}}\colon\ \mathcal{M}_{\mathbf{p}} \longrightarrow \mathcal{N}_{\Psi(\mathbf{p})}$$

given by

$$\Psi_{*\mathbf{p}}(\mathbf{v_p})[f] = \mathbf{v_p}[f\circ\Psi] \tag{24.11}$$

for each $f \in \mathfrak{F}(\mathcal{N})$ *and* $\mathbf{v_p} \in \mathcal{M}_{\mathbf{p}}$*.*

In order for this definition to make sense, we must be sure that the image of $\Psi_{*\mathbf{p}}$ is actually contained in the tangent space $\mathcal{N}_{\Psi(\mathbf{p})}$.

Lemma 24.30. *Let* $\Psi\colon \mathcal{M} \to \mathcal{N}$ *be a differentiable map, and let* $\mathbf{p} \in \mathcal{M}$*,* $\mathbf{v_p} \in \mathcal{M}_{\mathbf{p}}$*. Define*

$$\Psi_{*\mathbf{p}}(\mathbf{v_p})\colon\ \mathfrak{F}(\mathcal{N}) \longrightarrow \mathbb{R}$$

by (24.11)*. Then* $\Psi_{*\mathbf{p}}(\mathbf{v_p}) \in \mathcal{N}_{\Psi(\mathbf{p})}$*.*

Proof. For example, we show that $\Psi_{*\mathbf{p}}(\mathbf{v_p})$ is Leibnizian. Using the fact that $\mathbf{v_p}$ is Leibnizian, we have

$$\begin{aligned}\Psi_{*\mathbf{p}}(\mathbf{v_p})[fg] &= \mathbf{v_p}[(fg)\circ\Psi] = \mathbf{v_p}\big[(f\circ\Psi)(g\circ\Psi)\big]\\ &= (f\circ\Psi)(\mathbf{p})\mathbf{v_p}[g\circ\Psi] + (g\circ\Psi)(\mathbf{p})\mathbf{v_p}[f\circ\Psi]\\ &= f\big(\Psi(\mathbf{p})\big)\Psi_{*\mathbf{p}}(\mathbf{v_p})[g] + g\big(\Psi(\mathbf{p})\big)\Psi_{*\mathbf{p}}(\mathbf{v_p})[f]. \blacksquare\end{aligned}$$

The next results establish properties of tangent maps.

Lemma 24.31. *Suppose that $\mathcal{M}$ is an n-dimensional differentiable manifold and that $\mathbf{x}\colon \mathcal{U} \to \mathcal{M}$ is a patch, where $\mathcal{U}$ is an open subset of $\mathbb{R}^n$. Write $\mathbf{x}^{-1} = (x_1, \ldots, x_n)$. Then*

$$\mathbf{x}_{*\mathbf{q}}\left(\frac{\partial}{\partial u_i}\bigg|_{\mathbf{q}}\right) = \mathbf{x}_{u_i}(\mathbf{q}) = \frac{\partial}{\partial x_i}\bigg|_{\mathbf{x}(\mathbf{q})} \tag{24.12}$$

for $\mathbf{q} \in \mathcal{U}$ and $i = 1, \ldots, n$.

Proof. For $f \in \mathfrak{F}(\mathcal{M})$ we have by (24.3) that

$$\mathbf{x}_{*\mathbf{q}}\left(\frac{\partial}{\partial u_i}\bigg|_{\mathbf{q}}\right)[f] = \frac{\partial}{\partial u_i}\bigg|_{\mathbf{q}}[f\circ\mathbf{x}] = \frac{\partial f}{\partial x_i}\bigg|_{\mathbf{x}(\mathbf{q})} = \frac{\partial}{\partial x_i}\bigg|_{\mathbf{x}(\mathbf{q})}[f]. \blacksquare$$

The tangent map $\Psi_{*\mathbf{p}}$ is a kind of dual of the homomorphism $\mathfrak{F}(\mathcal{N}) \to \mathfrak{F}(\mathcal{M})$ defined by mapping a function f to $f \circ \Psi$.

Lemma 24.32. *Let $\Psi\colon \mathcal{M} \to \mathcal{N}$ be a differentiable map and $\mathbf{p} \in \mathcal{M}$.*

(i) *The map $\Psi_{*\mathbf{p}}\colon \mathcal{M}_{\mathbf{p}} \to \mathcal{N}_{\Psi(\mathbf{p})}$ is a linear map between vector spaces.*

(ii) *If $\mathcal{M} \xrightarrow{\Psi} \mathcal{N} \xrightarrow{\Phi} P$ are differentiable maps, then*

$$(\Phi\circ\Psi)_{*\mathbf{p}} = \Phi_{*\Psi(\mathbf{p})} \circ \Psi_{*\mathbf{p}}.$$

(iii) *Let $\mathbf{x}\colon \mathcal{U} \to \mathcal{M}$ be a patch on $\mathcal{M}$ at $\mathbf{p}$ and $\mathbf{y}\colon \mathcal{V} \to \mathcal{N}$ a patch on $\mathcal{N}$ at $\Psi(\mathbf{p})$. Write $\mathbf{x}^{-1} = (x_1, \ldots, x_m)$ and $\mathbf{y}^{-1} = (y_1, \ldots, y_n)$, where $m = \dim\mathcal{M}$ and $n = \dim\mathcal{N}$. Then*

$$\Psi_{*\mathbf{p}}\left(\frac{\partial}{\partial x_j}\bigg|_{\mathbf{p}}\right) = \sum_{i=1}^{n} \frac{\partial(y_i\circ\Psi)}{\partial x_j}(\mathbf{p})\frac{\partial}{\partial y_i}\bigg|_{\Psi(\mathbf{p})} \tag{24.13}$$

for $j = 1, \ldots, m$.

Proof. The proofs of (i) and (ii) are straightforward. To prove (iii), we use the basis theorem (Theorem 24.25). Since

$$\Psi_{*\mathbf{p}}\left(\frac{\partial}{\partial x_j}\bigg|_{\mathbf{p}}\right) \in \mathcal{N}_{\Psi(\mathbf{p})},$$

and

$$\left\{\frac{\partial}{\partial y_1}\bigg|_{\Psi(\mathbf{p})}, \dots, \frac{\partial}{\partial y_n}\bigg|_{\Psi(\mathbf{p})}\right\}$$

is a basis for $\mathcal{N}_{\Psi(\mathbf{p})}$, we can write

$$\Psi_{*\mathbf{p}}\left(\frac{\partial}{\partial x_j}\bigg|_{\mathbf{p}}\right) = \sum_{i=1}^{n} a_{ij}\frac{\partial}{\partial y_i}\bigg|_{\Psi(\mathbf{p})}. \tag{24.14}$$

To compute the a_{ij}'s, we apply both sides of (24.14) to y_k. We have

$$\left(\sum_{i=1}^{n} a_{ij}\frac{\partial}{\partial y_i}\bigg|_{\Psi(\mathbf{p})}\right)[y_k] = a_{kj}$$

while

$$\Psi_{*\mathbf{p}}\left(\frac{\partial}{\partial x_j}\bigg|_{\mathbf{p}}\right)[y_k] = \left(\frac{\partial}{\partial x_j}\bigg|_{\mathbf{p}}\right)[y_k \circ \Psi] = \frac{\partial(y_k \circ \Psi)}{\partial x_j}(\mathbf{p}).$$

Hence

$$a_{ij} = \frac{\partial(y_i \circ \Psi)}{\partial x_j}(\mathbf{p}),$$

and we get (24.13). ▮

Definition 24.33. *Let $\Psi\colon \mathcal{M} \to \mathcal{N}$ be a differentiable map, where $\mathcal{M}, \mathcal{N}$ are differentiable manifolds of dimensions m, n respectively. Let $\mathbf{p} \in \mathcal{M}$, let $(\mathbf{x}, \mathcal{U})$ be a patch on $\mathcal{M}$ at $\mathbf{p}$ and $(\mathbf{y}, \mathcal{V})$ a patch on $\mathcal{N}$ at $\Psi(\mathbf{p})$. The* ***Jacobian matrix*** *$\mathscr{J}(\Psi)(\mathbf{p})$ of Ψ at $\mathbf{p}$ relative to $\mathbf{x}$ and $\mathbf{y}$ is the matrix of $\Psi_{*\mathbf{p}}$ relative to the bases*

$$\left\{\frac{\partial}{\partial x_1}\bigg|_{\mathbf{p}}, \dots, \frac{\partial}{\partial x_m}\bigg|_{\mathbf{p}}\right\} \quad \text{and} \quad \left\{\frac{\partial}{\partial y_1}\bigg|_{\Psi(\mathbf{p})}, \dots, \frac{\partial}{\partial y_n}\bigg|_{\Psi(\mathbf{p})}\right\}.$$

Explicitly, $\mathscr{J}(\Psi)(\mathbf{p})$ is the matrix $\left(\dfrac{\partial(y_i \circ \Psi)}{\partial x_j}(\mathbf{p})\right)$.

We can now get the manifold formulation of the inverse function theorem from multivariable calculus:

Theorem 24.34. (Inverse Function Theorem) *Let $\Psi\colon \mathcal{M} \to \mathcal{N}$ be a differentiable map, and let $\mathbf{p} \in \mathcal{M}$. Suppose $\dim \mathcal{M} = \dim \mathcal{N} = n$. Then the following conditions are equivalent:*

(i) $\Psi_{*\mathbf{p}}$ *is a linear isomorphism*;

(ii) *the Jacobian matrix* $\mathscr{J}(\Psi)(\mathbf{p})$ *is invertible*;

(iii) *there exist neighborhoods* $\mathcal{P}$ *of* $\mathbf{p}$ *and* $\mathcal{Q}$ *of* $\Psi(\mathbf{p})$ *such that the restriction* $\Psi|\mathcal{P}\colon \mathcal{P} \to \mathcal{Q}$ *has a differentiable inverse* $(\Psi|\mathcal{P})^{-1}\colon \mathcal{Q} \to \mathcal{P}$.

Proof. That (i) implies (ii) is a standard fact of linear algebra. Also, it is easy to see from Lemma 24.32(ii) that (iii) implies (i).

To show that (ii) implies (iii), consider $\mathbf{y}^{-1} \circ \Psi \circ \mathbf{x}$, where $(\mathbf{x},\mathcal{U})$ is a patch on $\mathcal{M}$ with $\mathbf{p} \in \mathbf{x}(\mathcal{U})$ and $(\mathbf{y},\mathcal{V})$ is a patch on $\mathcal{N}$ with $\Psi(\mathbf{p}) \in \mathbf{y}(\mathcal{V})$. Without loss of generality, we can assume that $\mathbf{p} = \mathbf{x}(\mathbf{0})$ and $\Psi(\mathbf{p}) = \mathbf{y}(\mathbf{0})$ so that

$$(\mathbf{y}^{-1} \circ \Psi \circ \mathbf{x})(\mathbf{0}) = \mathbf{0}.$$

Write $\mathbf{x}^{-1} = (x_1, \ldots, x_n)$ and $\mathbf{y}^{-1} = (y_1, \ldots, y_n)$. By assumption

$$\det\left(\frac{\partial(u_i \circ \mathbf{y}^{-1} \circ \Psi \circ \mathbf{x})}{\partial u_j}(\mathbf{0})\right) = \det\left(\frac{\partial(y_i \circ \Psi)}{\partial x_j}(\mathbf{p})\right) = \det \mathscr{J}(\Psi)(\mathbf{p}) \neq 0.$$

The inverse function theorem for $\mathbb{R}^n$ implies that $\mathbf{y}^{-1} \circ \Psi \circ \mathbf{x}$ has a local inverse. Hence so does Ψ. ▮

Corollary 24.35. *Let* $(x_1, \ldots, x_n)$ *be a coordinate system for* $\mathcal{M}$ *at* $\mathbf{p}$, *where each* x_j *is defined on a neighborhood* $\mathcal{W}$ *of* $\mathbf{p}$. *Then functions* $f_1, \ldots, f_n \in \mathfrak{F}(\mathcal{W})$ *form a coordinate system for* $\mathcal{M}$ *at* $\mathbf{p}$ *if and only if*

$$\det\left(\frac{\partial f_i}{\partial x_j}\right) \neq 0$$

on $\mathcal{W}$.

Definition 24.36. *A differentiable map* $\Psi\colon \mathcal{M} \to \mathcal{N}$ *is called* ***regular***, *provided* $\Psi_{*\mathbf{p}}$ *is a nonsingular linear transformation for each* $\mathbf{p} \in \mathcal{M}$.

Corollary 24.37. *A map* $\Psi\colon \mathcal{M} \to \mathcal{N}$ *is a local diffeomorphism if and only if each* $\Psi_{*\mathbf{p}}$ *is a linear isomorphism.*

Next, we discuss the notion of curve in a differentiable manifold using the concepts we have developed. In what follows I denotes an open interval of the real line $\mathbb{R}$ and d/du the natural coordinate vector field on it. For each $t \in I$ we have a canonical element of the tangent space $\mathbb{R}_t$ to $\mathbb{R}$ at the point t, namely

$$\left.\frac{d}{du}\right|_t.$$

Definition 24.38. *A* ***curve*** *on a differentiable manifold $\mathcal{M}$ is a differentiable function $\boldsymbol{\alpha}\colon I \to \mathcal{M}$. If $t \in I$ the* ***velocity vector*** *of $\boldsymbol{\alpha}$ at t is the tangent vector*

$$\boldsymbol{\alpha}'(t) = \boldsymbol{\alpha}_{*t}\left(\frac{d}{du}\bigg|_t\right) \in \mathcal{M}_{\boldsymbol{\alpha}(t)}.$$

We do not exclude the possibility that the interval I is half-infinite or all of $\mathbb{R}$. One can usually arrange to have zero in the domain I of a curve $\boldsymbol{\alpha}$; it is a convenient reference point. A curve on a surface in $\mathbb{R}^3$ is also a curve in the sense of this definition and the two notions of velocity vector are equivalent. We shall say that a curve $\boldsymbol{\alpha}$ ***starts*** at $\mathbf{p} \in \mathcal{M}$ provided that $\boldsymbol{\alpha}(0) = \mathbf{p}$, and that $\boldsymbol{\alpha}$ has ***initial velocity*** $\mathbf{v_p} \in \mathcal{M}_\mathbf{p}$, provided $\boldsymbol{\alpha}'(0) = \mathbf{v_p}$. We leave the reader to reconcile the notation $\boldsymbol{\alpha}'(t)$ in Definition 24.38 with its usual meaning when $\boldsymbol{\alpha}$ is merely a differentiable mapping $\mathbb{R} \to \mathbb{R}$.

Lemma 24.39. *Let $\boldsymbol{\alpha}\colon I \to \mathcal{M}$ be a curve. Then the value of the velocity vector $\boldsymbol{\alpha}'(t)$ on a function $f \in \mathfrak{F}(\mathcal{M})$ is given by*

$$\boldsymbol{\alpha}'(t)[f] = \frac{d(f \circ \boldsymbol{\alpha})}{du}(t).$$

Proof. We have

$$\boldsymbol{\alpha}'(t)[f] = \boldsymbol{\alpha}_{*t}\left(\frac{d}{du}\bigg|_t\right)[f] = \frac{d(f \circ \boldsymbol{\alpha})}{du}(t). \quad \blacksquare$$

Lemma 24.39 makes rigorous the earlier suggestion that for a tangent vector $\mathbf{v_p}$ the number $\mathbf{v_p}[f]$ is the derivative of f in the direction $\mathbf{v_p}$. This is because for any curve $\boldsymbol{\alpha}$ with initial velocity $\mathbf{v_p}$ we have

$$\mathbf{v_p}[f] = \frac{d(f \circ \boldsymbol{\alpha})}{du}(0).$$

Lemma 24.40. *Let $\boldsymbol{\alpha}$ be a curve in $\mathcal{M}$, and let $(x_1, \ldots, x_n)$ be a coordinate system at $\boldsymbol{\alpha}(t) \in \mathcal{M}$. Then*

$$\boldsymbol{\alpha}'(t) = \sum_{i=1}^{n} \frac{d(x_i \circ \boldsymbol{\alpha})}{du}(t)\frac{\partial}{\partial x_i}\bigg|_{\boldsymbol{\alpha}(t)}.$$

Proof. This is an immediate consequence of Theorem 24.25. $\blacksquare$

A definition of the acceleration vector of a curve in an abstract manifold requires a *connection* on the manifold, and is given in Section 25.1.

Next, we prove the generalization to differentiable manifolds of Lemma 1.9 on page 7. Although the following lemma can be proved by showing that it is a special case of Lemma 24.27, it is easier to prove it directly.

Lemma 24.41. (The Chain Rule for curves on a manifold) *Let* $\boldsymbol{\alpha}\colon (a,b) \to \mathcal{M}$ *and* $h\colon (c,d) \to (a,b)$ *be differentiable. Put* $\boldsymbol{\beta} = \boldsymbol{\alpha} \circ h$. *Then*

$$\boldsymbol{\beta}' = (\boldsymbol{\alpha}' \circ h)\, h'.$$

Proof. We have

$$\begin{aligned}\boldsymbol{\beta}'(t) \;=\; \boldsymbol{\beta}_{*t}\left(\frac{d}{du}\bigg|_t\right) \;&=\; \boldsymbol{\alpha}_{*h(t)} \circ h_{*t}\left(\frac{d}{du}\bigg|_t\right)\\ &=\; \boldsymbol{\alpha}_{*h(t)}\left(h'(t)\frac{d}{du}\bigg|_{h(t)}\right) = h'(t)\,\boldsymbol{\alpha}'\big(h(t)\big). \quad \blacksquare\end{aligned}$$

24.5 Vector Fields on Manifolds

The notion of derivation of an algebra comes up frequently in differential geometry. Let us give the general definition and then specialize it to show that a vector field is a derivation.

Definition 24.42. *Let* A *be an algebra* (*which is not necessarily associative nor commutative*) *over a field* $\mathbb{F}$. *A* ***derivation*** *of* A *is a mapping* $D\colon A \to A$ *such that*

$$\begin{cases} D(af+bg) \;=\; a\,Df + b\,Dg,\\ \qquad D(fg) \;=\; f(Dg) + (Df)g \end{cases}$$

for all $a, b \in \mathbb{F}$ *and* $f, g \in A$.

Shortly, we shall define a vector field as a derivation of a special sort. But first let us study derivations in general.

Definition 24.43. *If* D_1 *and* D_2 *are derivations of an algebra, then the* ***bracket*** $[D_1, D_2]$ *of* D_1 *and* D_2 *is defined by*

$$[D_1, D_2] = D_1 D_2 - D_2 D_1.$$

The proofs of the next two lemmas are easy.

Lemma 24.44. *If* D_1 *and* D_2 *are derivations of an algebra, then so are* $[D_1, D_2]$, $D_1 + D_2$ *and* aD_1 *for* $a \in \mathbb{F}$.

Lemma 24.45. *Let* D_1, D_2 *and* D_3 *be derivations of an algebra. Then*:

(i) $[D_1, D_2] = -[D_2, D_1]$.

(ii) *The* Jacobi identity *holds*:

$$\big[[D_1, D_2], D_3\big] + \big[[D_3, D_1], D_2\big] + \big[[D_2, D_3], D_1\big] = 0.$$

We sometimes abbreviate the Jacobi identity to

$$\mathop{\mathfrak{S}}_{1,2,3} \big[[D_1, D_2], D_3\big] = 0,$$

where $\mathfrak{S}$ denotes the cyclic sum. It is an important ingredient in

Definition 24.46. *Let $\mathbb{F}$ be a field. A **Lie algebra** over $\mathbb{F}$ is a vector space V over $\mathbb{F}$ with a bracket $[\ ,\ ]\colon V \times V \to V$ such that*

$$\begin{cases} [X, X] = 0, \\ [aX + bY, Z] = a[X, Z] + b[Y, Z], \\ [Z, aX + bY] = a[Z, X] + b[Z, Y], \\ \mathop{\mathfrak{S}}_{X,Y,Z} \big[X, [Y, Z]\big] = 0 \end{cases}$$

for each $X, Y, Z \in V$ and $a, b \in \mathbb{F}$.

We have a ready-made example of a Lie algebra.

Lemma 24.47. *Let A be a commutative algebra over a field $\mathbb{F}$ and denote by $\mathfrak{D}(A)$ the set of derivations of A. Then $\mathfrak{D}(A)$ is a module over A and a Lie algebra over $\mathbb{F}$.*

Derivations arise in many different contexts in differential geometry, and the one of immediate interest provides us with an efficient way to extend the definition of a vector field on $\mathbb{R}^n$, given in Chapter 9, to manifolds.

Definition 24.48. *A **vector field** $\mathbf{X}$ on a differentiable manifold $\mathcal{M}$ is a derivation of the algebra $\mathfrak{F}(\mathcal{M})$ of real-valued differentiable functions on $\mathcal{M}$.*

Thus a vector field $\mathbf{X}$ on $\mathcal{M}$ is a mapping $\mathbf{X}\colon \mathfrak{F}(\mathcal{M}) \to \mathfrak{F}(\mathcal{M})$ satisfying

$$\mathbf{X}[af + bg] = a\mathbf{X}[f] + b\mathbf{X}[g] \qquad \text{(the linearity property)},$$
$$\mathbf{X}[fg] = f\mathbf{X}[g] + g\mathbf{X}[f] \qquad \text{(the Leibnizian property)},$$

for $a, b \in \mathbb{R}$ and $f, g \in \mathfrak{F}(\mathcal{M})$. Also, the ***bracket*** $[\mathbf{X}, \mathbf{Y}]$ of vector fields $\mathbf{X}$ and $\mathbf{Y}$ is given by

$$[\mathbf{X}, \mathbf{Y}] = \mathbf{XY} - \mathbf{YX}.$$

Note that $\mathbf{XY}$ (defined by $\mathbf{XY}[f] = \mathbf{X}(\mathbf{Y}[f])$) is a well-defined operator on $\mathfrak{F}(\mathcal{M})$, but not a vector field because the Leibnizian property is not satisfied. We put

$$\mathfrak{X}(\mathcal{M}) = \{\, \mathbf{X} \mid \mathbf{X} \text{ is a vector field on } \mathcal{M} \,\}.$$

Lemma 24.47 tells us that $\mathfrak{X}(\mathcal{M})$ is a module over $\mathfrak{F}(\mathcal{M})$ and a Lie algebra over $\mathbb{R}$. In other words, we may multiply vector fields by either real numbers or real-valued functions. Moreover, we have a bracket 'multiplication' between vector fields that yields another vector field in $\mathfrak{X}(\mathcal{M})$. The bracket is *not* multilinear with respect to functions; instead, there is a more complicated rule:

Lemma 24.49. *Let $\mathbf{X}, \mathbf{Y} \in \mathfrak{X}(\mathcal{M})$ and $f, g \in \mathfrak{F}(\mathcal{M})$. Then*

$$[f\mathbf{X},\ g\mathbf{Y}] = fg[\mathbf{X}, \mathbf{Y}] + f\mathbf{X}[g]\mathbf{Y} - g\mathbf{Y}[f]\mathbf{X}. \tag{24.15}$$

The definition of vector field that we have given is very abstract. Probably a more intuitive notion of vector field is that of a function that assigns to each point $\mathbf{p}$ of a manifold $\mathcal{M}$ a tangent vector $\mathbf{X_p}$. We now show that this intuitive notion is equivalent to the definition that we have given. First, we introduce

Definition 24.50. *The* ***tangent bundle*** *of a differentiable manifold $\mathcal{M}$ is the set*

$$T(\mathcal{M}) = \left\{ (\mathbf{p}, \mathbf{v_p}) \ \middle|\ \mathbf{v_p} \in \mathcal{M}_\mathbf{p},\ \ \mathbf{p} \in \mathcal{M} \right\}.$$

This object is both an important source of new manifolds, and a means of interpreting the definition of vector field.

Theorem 24.51. *The tangent bundle $T(\mathcal{M})$ of a differentiable manifold $\mathcal{M}$ is naturally a differentiable manifold whose dimension is twice the dimension of $\mathcal{M}$. Furthermore, the* ***projection map*** *$\pi\colon T(\mathcal{M}) \to \mathcal{M}$ defined by $\pi(\mathbf{p}, \mathbf{v_p}) = \mathbf{p}$ is a differentiable map.*

Lemma 24.52. *Associated with each vector field $\mathbf{X} \in \mathfrak{X}(\mathcal{M})$ is a* ***section*** *of $T(\mathcal{M})$, that is, a differentiable map $\mathbf{X}\colon \mathcal{M} \to T(\mathcal{M})$ such that $\pi \circ \mathbf{X} = \mathbf{1}_\mathcal{M}$, where $\mathbf{1}_\mathcal{M}$ denotes the identity map on $\mathcal{M}$. In particular, $\mathbf{X}$ is a map that associates a tangent vector $\mathbf{X_p} \in \mathcal{M}_\mathbf{p}$ with each point $\mathbf{p} \in \mathcal{M}$.*

Conversely, each section of the tangent bundle gives rise in a natural way to an element of $\mathfrak{X}(\mathcal{M})$.

From now on we consider vector fields as derivations of $\mathfrak{F}(\mathcal{M})$, or as sections of $T(\mathcal{M})$, whichever seems more convenient. If $\mathbf{X} \in \mathfrak{X}(\mathcal{M})$, we write $\mathbf{X_p}$ for the tangent vector in $\mathcal{M}_\mathbf{p}$ determined by $\mathbf{X}$.

Definition 24.53. *Let $(x_1, \ldots, x_n)$ be a system of local coordinates for $\mathcal{M}$ at $\mathbf{p}$ defined on an open set $\mathcal{W} \subset \mathcal{M}$. Then*

$$\frac{\partial}{\partial x_i}\colon \mathfrak{F}(\mathcal{W}) \longrightarrow \mathfrak{F}(\mathcal{W})$$

is the operator that assigns to each function $f \in \mathfrak{F}(\mathcal{W})$ its partial derivative $\partial f/\partial x_i$.

The next lemma is an obvious consequence of Lemma 24.20.

Lemma 24.54. *For each $i = 1, \ldots, n$, and $\mathbf{p} \in \mathcal{W}$, we have*

$$\frac{\partial}{\partial x_i} \in \mathfrak{X}(\mathcal{W}) \qquad \text{and} \qquad \left.\frac{\partial}{\partial x_i}\right|_{\mathbf{p}} \in \mathcal{M}_{\mathbf{p}}.$$

We are now in a position to give a version of the Basis Theorem on page 823 for vector fields.

Corollary 24.55. *Let $\mathbf{X} \in \mathfrak{X}(\mathcal{M})$ and let $\mathbf{x}^{-1} = (x_1, \ldots, x_n)$ be a system of local coordinates defined on an open set $\mathcal{W}$. Then $\mathbf{X}$ can be written as*

$$\mathbf{X} = \sum_{i=1}^{n} \mathbf{X}[x_i] \frac{\partial}{\partial x_i} \tag{24.16}$$

on $\mathcal{W}$.

Proof. It follows from Theorem 24.25 that

$$\mathbf{X}_{\mathbf{p}} = \sum_{i=1}^{n} \mathbf{X}_{\mathbf{p}}[x_i] \left.\frac{\partial}{\partial x_i}\right|_{\mathbf{p}}. \tag{24.17}$$

Since (24.17) holds for all $\mathbf{p} \in \mathcal{W}$, we get (24.16). ∎

We now consider the effect of a differentiable map $\Psi \colon \mathcal{M} \to \mathcal{N}$ on vector fields. Unfortunately, it is not always possible to transfer vector fields on $\mathcal{M}$ to vector fields on $\mathcal{N}$. The problem is that if $\mathbf{X} \in \mathfrak{X}(\mathcal{M})$ and $\mathbf{p}, \mathbf{q} \in \mathcal{M}$ are points such that $\Psi(\mathbf{p}) = \Psi(\mathbf{q})$, we may have that

$$\Psi_{*\mathbf{p}}(\mathbf{X}_{\mathbf{p}}) \neq \Psi_{*\mathbf{q}}(\mathbf{X}_{\mathbf{q}}).$$

An example of the phenomenon occurs for the vector field $\mathbf{X}$ in $\mathbb{R}^3$ defined by

$$\mathbf{X}_{(x,y,z)} = (-y,\, x,\, 0),$$

illustrated in Figure 24.3. The projection $(x, y, z) \mapsto (y, z)$ to the yz-plane 'muddles up' the vectors. By contrast, mapping (x, y, z) to (x, y) gives rise to an unambiguous vector field in the xy-plane, and corresponds to the favorable case formulated next.

Definition 24.56. *Let $\Psi \colon \mathcal{M} \to \mathcal{N}$ be a differentiable map, and let $\mathbf{X} \in \mathfrak{X}(\mathcal{M})$, $\mathbf{Y} \in \mathfrak{X}(\mathcal{N})$. We say that $\mathbf{X}$ and $\mathbf{Y}$ are **Ψ-related** provided that*

$$\Psi_{*\mathbf{p}}(\mathbf{X}_{\mathbf{p}}) = \mathbf{Y}_{\Psi(\mathbf{p})}$$

for all $\mathbf{p} \in \mathcal{M}$.

We use the notation $\mathbf{X}^{\Psi} = \mathbf{Y}$ or $\Psi_*(\mathbf{X}) = \mathbf{Y}$, but beware that $\mathbf{Y}$ is not in general determined by $\mathbf{X}$ if Ψ is not surjective.

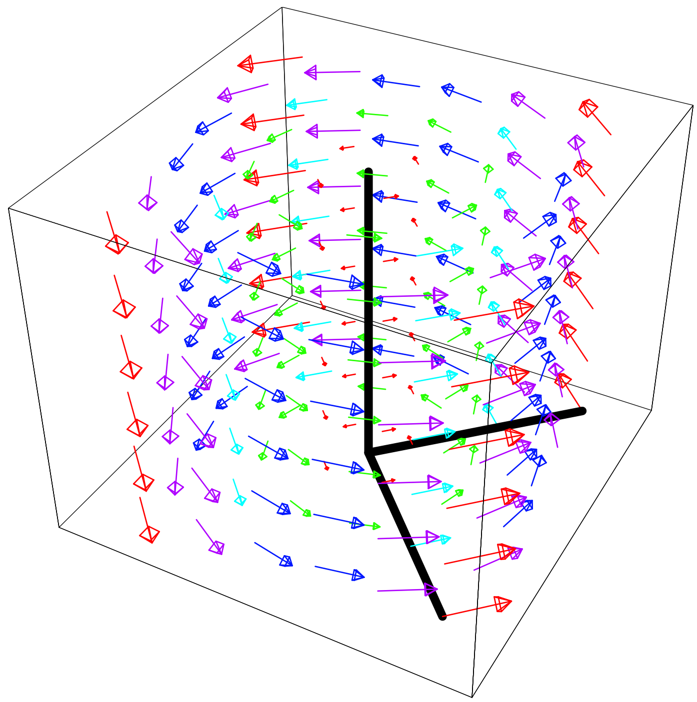


Figure 24.3: A swirling vector field in $\mathbb{R}^3$

The proof of the following results is straightforward.

Lemma 24.57. *Let $\Psi\colon \mathcal{M} \to \mathcal{N}$ be a differentiable map.*

(i) *A vector field $\mathbf{X} \in \mathfrak{X}(\mathcal{M})$ is Ψ-related to a vector field $\mathbf{Y} \in \mathfrak{X}(\mathcal{N})$ if and only if*

$$\mathbf{Y}[f] \circ \Psi = \mathbf{X}[f \circ \Psi]$$

for all $f \in \mathfrak{F}(\mathcal{N})$.

(ii) *Let $\mathbf{W}, \mathbf{X} \in \mathfrak{X}(\mathcal{M})$ and $\mathbf{Y}, \mathbf{Z} \in \mathfrak{X}(\mathcal{N})$ with $\mathbf{W}^\Psi = \mathbf{Y}$ and $\mathbf{X}^\Psi = \mathbf{Z}$. Then $[\mathbf{W}, \mathbf{X}]$ is Ψ-related to $[\mathbf{Y}, \mathbf{Z}]$, so*

$$[\mathbf{W}^\Psi, \mathbf{X}^\Psi] = [\mathbf{Y}, \mathbf{Z}] = [\mathbf{W}, \mathbf{X}]^\Psi.$$

(iii) *If $\Psi\colon \mathcal{M} \to \mathcal{N}$ is a diffeomorphism, then for any $\mathbf{X} \in \mathfrak{X}(\mathcal{M})$ there exists a unique $\mathbf{Y} \in \mathfrak{X}(\mathcal{N})$ such that $\mathbf{X}^\Psi = \mathbf{Y}$.*

The vector-field version of Lemma 24.31 is

$$\mathbf{x}_*\Big(\frac{\partial}{\partial u_i}\Big) = \frac{\partial}{\partial x_i} = \mathbf{x}_{u_i} \circ \mathbf{x}^{-1}. \tag{24.18}$$

From it, we may deduce an important vanishing result:

Corollary 24.58. *Let $\mathcal{M}$ be an n-dimensional differentiable manifold and let $\mathbf{x}\colon \mathcal{U} \to \mathcal{M}$ be a patch, where $\mathcal{U}$ is an open subset of $\mathbb{R}^n$. Then*

$$\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = \left[\mathbf{x}_{u_i}, \mathbf{x}_{u_j}\right] = 0,$$

for $1 \leqslant i, j \leqslant n$.

Proof. We have

$$\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = \left[\mathbf{x}_*\left(\frac{\partial}{\partial u_i}\right), \mathbf{x}_*\left(\frac{\partial}{\partial u_j}\right)\right] = \mathbf{x}_*\left(\left[\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\right]\right) = 0,$$

proving the second equation. ▮

24.6 Tensor Fields

It will also be necessary to consider *tensor fields.* Tensor analysis was developed by Ricci-Curbastro[4] (see [Ricci]). The modern invariant notation described below was introduced by Koszul [Kosz].

Instead of developing the most general situation, it will be sufficient for our purposes to treat two special cases.

Definition 24.59. *A* ***covariant tensor field*** *of* ***degree*** *r on a differentiable manifold $\mathcal{M}$ is a mapping*

$$\alpha\colon \underbrace{\mathfrak{X}(\mathcal{M}) \times \cdots \times \mathfrak{X}(\mathcal{M})}_{r \text{ times}} \longrightarrow \mathfrak{F}(\mathcal{M})$$

that satisfies

$$\alpha(\mathbf{X}_1, \ldots, f_i\mathbf{Y}_i + g_i\mathbf{Z}_i, \ldots, \mathbf{X}_r) = f_i\,\alpha(\mathbf{X}_1, \ldots, \mathbf{Y}_i, \ldots, \mathbf{X}_r) + g_i\,\alpha(\mathbf{X}_1, \ldots, \mathbf{Z}_i, \ldots, \mathbf{X}_r) \tag{24.19}$$

for all $f_i, g_i \in \mathfrak{F}(\mathcal{M})$, $\mathbf{X}_1, \ldots, \mathbf{X}_r$, $\mathbf{Y}_i, \mathbf{Z}_i \in \mathfrak{X}(\mathcal{M})$, and for each index i in turn. A ***vectorvariant tensor field*** *of* ***degree*** *r is a mapping*

$$\Phi\colon \underbrace{\mathfrak{X}(\mathcal{M}) \times \cdots \times \mathfrak{X}(\mathcal{M})}_{r \text{ times}} \longrightarrow \mathfrak{X}(\mathcal{M})$$

that satisfies (24.19).

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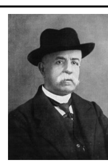
[4] 

Gregorio Ricci-Curbastro (1853–1925). Italian mathematician, professor at the University of Padua. He invented the absolute differential calculus between 1884 and 1894. It became the foundation of the tensor analysis that was used by Einstein in his theory of general relativity (see [HaEl]).

Remarks

(1) Condition (24.19) asserts that α and Φ are *multilinear with respect to functions*. On the other hand, the Lie bracket [,] is not a tensor field because it is not multilinear with respect to functions; instead it satisfies the more complicated formula (24.15).

(2) Sometimes a covariant tensor field of degree r is said to be a tensor field of type $(r, 0)$, and a vectorvariant tensor field of degree r is said to be a tensor field of type $(r, 1)$. It is also possible to define tensor fields of type (r, s), but we ignore the case $s \geqslant 2$.

(3) There is actually a definition in linear algebra that is very similar to the definition of tensor field. Applying it to the tangent space at a point $\mathbf{p}$ to a differentiable manifold $\mathcal{M}$ gives

Definition 24.60. *A* ***covariant tensor*** *at* $\mathbf{p} \in \mathcal{M}$ *is a multilinear map*

$$\alpha \colon \underbrace{\mathcal{M}_{\mathbf{p}} \times \cdots \times \mathcal{M}_{\mathbf{p}}}_{r \text{ times}} \longrightarrow \mathbb{R}$$

that satisfies

$$\alpha(\mathbf{x}_1, \ldots, a_i\mathbf{y}_i + b_i\mathbf{z}_i, \ldots, \mathbf{x}_r) = a_i\alpha(\mathbf{x}_1, \ldots, \mathbf{y}_i, \ldots, \mathbf{x}_r) + b_i\alpha(\mathbf{x}_1, \ldots, \mathbf{z}_i, \ldots, \mathbf{x}_r)$$

for all $a_i, b_i \in \mathbb{R}$*, and all* $\mathbf{x}_1, \ldots, \mathbf{x}_r$, $\mathbf{y}_i, \mathbf{z}_i \in \mathcal{M}_{\mathbf{p}}$. *A* ***vectorvariant tensor*** *at* $\mathbf{x}$ *is defined similarly, by replacing* $\mathbb{R}$ *with* $\mathcal{M}_{\mathbf{p}}$.

On page 834 we noted that some differentiable mappings between manifolds $\mathcal{M}$ and $\mathcal{N}$ do not always induce a map between the algebras of vector fields of $\mathcal{M}$ and $\mathcal{N}$. The situation is much better with covariant tensor fields.

Definition 24.61. *Let* $\Psi \colon \mathcal{M} \to \mathcal{N}$ *be a differentiable mapping, and let* α *be a covariant tensor field of degree* r *on* $\mathcal{N}$. *Then the* ***pullback*** *of* α *is the covariant tensor field* $\Psi^*(\alpha)$ *on* $\mathcal{M}$ *given by*

$$\Psi^*(\alpha)(\mathbf{v}_1, \ldots, \mathbf{v}_r) = \alpha\big(\Psi_{*p}(\mathbf{v}_1), \ldots, \Psi_{*p}(\mathbf{v}_r)\big),$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_r$ *are arbitrary tangent vectors at an arbitrary point* $\mathbf{p} \in \mathcal{M}$.

The following lemma is easy to prove.

Lemma 24.62. *Let* $\Psi \colon \mathcal{M} \to \mathcal{N}$ *be a differentiable mapping, and let* α *and* β *be covariant tensor fields of degree* r *on* $\mathcal{N}$. *Also, let* $f, g \in \mathfrak{F}(\mathcal{N})$. *Then*

$$\Psi^*(f\alpha + g\beta) = (f \circ \Psi)\,\Psi^*(\alpha) + (g \circ \Psi)\,\Psi^*(\beta).$$

Let us single out one of the simplest types of covariant tensor fields.

Definition 24.63. *A **differential** 1**-form** on a manifold $\mathcal{M}$ is an $\mathfrak{F}(\mathcal{M})$-linear map*

$$\omega\colon \mathfrak{X}(\mathcal{M}) \longrightarrow \mathfrak{F}(\mathcal{M}).$$

In other words, a 1*-form* ω *has the property that*

$$\omega(f\,\mathbf{X} + g\,\mathbf{Y}) = f\,\omega(\mathbf{X}) + g\,\omega(\mathbf{Y})$$

for $f, g \in \mathfrak{F}(\mathcal{M})$ *and* $\mathbf{X}, \mathbf{Y} \in \mathfrak{X}(\mathcal{M})$*. We denote the collection of* 1*-forms on* $\mathcal{M}$ *by* $\mathfrak{X}(\mathcal{M})^*$*, and we make it into a module over* $\mathfrak{F}(\mathcal{M})$ *by defining*

$$(f\,\omega + g\,\theta)(\mathbf{X}) = f\,\omega(\mathbf{X}) + g\,\theta(\mathbf{X})$$

for $f, g \in \mathfrak{F}(\mathcal{M})$*,* $\omega, \theta \in \mathfrak{X}(\mathcal{M})^*$ *and* $\mathbf{X} \in \mathfrak{X}(\mathcal{M})$*.*

If V is any vector space over any field $\mathbb{F}$ (in our case, almost invariably $\mathbb{R}$), its dual space is denoted by V^*. It is the new vector space

$$V^* = \{\, \alpha\colon V \to \mathbb{F} \;|\; \alpha \text{ is linear} \,\}$$

consisting of linear functionals, or linear mappings from V to the field. We see that Definition 24.63 extends this concept to modules, so that $\mathfrak{X}(\mathcal{M})^*$ is the dual module of $\mathfrak{X}(\mathcal{M})$.

Lemma 24.64. *Suppose* $\mathbf{X} \in \mathfrak{X}(\mathcal{M})$ *and* $\mathbf{p} \in \mathcal{M}$ *are such that* $\mathbf{X}_{\mathbf{p}} = 0$*. Then*

$$\omega(\mathbf{X})(\mathbf{p}) = 0$$

for all $\omega \in \mathfrak{X}(\mathcal{M})^*$*.*

Proof. Let $(\mathbf{x}, \mathcal{U})$ be a patch on $\mathcal{M}$ at $\mathbf{p}$ with $\mathbf{x}^{-1} = (x_1, \ldots, x_n)$. Using Corollary 24.55, we have

$$\mathbf{X} = \sum_{i=1}^{n} \mathbf{X}[x_i] \frac{\partial}{\partial x_i}$$

near $\mathbf{p}$. Also, $\mathbf{X}_{\mathbf{p}} = 0$ implies that $\mathbf{X}[x_i](\mathbf{p}) = 0$ for $i = 1, \ldots, n$. Hence,

$$\omega(\mathbf{X})(\mathbf{p}) = \omega\left(\sum_{i=1}^{n} \mathbf{X}[x_i] \frac{\partial}{\partial x_i}\right)(\mathbf{p}) = \sum_{i=1}^{n} \mathbf{X}[x_i](\mathbf{p})\,\omega\left(\frac{\partial}{\partial x_i}\right)(\mathbf{p}) = 0. \blacksquare$$

We can evaluate 1-forms at a point in the same way that we evaluate vector fields at a point.

Definition 24.65. *Let* $\omega \in \mathfrak{X}(\mathcal{M})^*$ *and* $\mathbf{p} \in \mathcal{M}$*. Then* $\omega_{\mathbf{p}}\colon \mathcal{M}_{\mathbf{p}} \to \mathbb{R}$ *is defined by*

$$\omega_{\mathbf{p}}(\mathbf{v}) = \omega(\mathbf{X})(\mathbf{p}),$$

where $\mathbf{X} \in \mathfrak{X}(\mathcal{M})$ *is any vector field for which* $\mathbf{X}_{\mathbf{p}} = \mathbf{v}$*.*

Lemma 24.66. *The definition of* $\omega_{\mathbf{p}}$ *does not depend on the choice of the vector field* $\mathbf{X}$.

Proof. Let $\mathbf{X}$ and $\widetilde{\mathbf{X}}$ be vector fields such that $\mathbf{X}_{\mathbf{p}} = \widetilde{\mathbf{X}}_{\mathbf{p}} = \mathbf{v}$. Then $\mathbf{X} - \widetilde{\mathbf{X}}$ vanishes at $\mathbf{p}$ so by Lemma 24.64,

$$\omega(\mathbf{X})(\mathbf{p}) - \omega(\widetilde{\mathbf{X}})(\mathbf{p}) = \omega(\mathbf{X} - \widetilde{\mathbf{X}})(\mathbf{p}) = 0. \blacksquare$$

There is a particularly important kind of 1-form.

Definition 24.67. *For* $f \in \mathfrak{F}(\mathcal{M})$ *the* ***differential*** df *of* f *is the* 1*-form defined by*

$$df(\mathbf{X}) = \mathbf{X}[f]$$

for $\mathbf{X} \in \mathfrak{X}(\mathcal{M})$.

We conclude this chapter with

Lemma 24.68. *Let* $(x_1, \ldots, x_n)$ *be a system of coordinates defined on an open set* $\mathcal{W} \subset \mathcal{M}$. *Then*

(i) *For each* $\mathbf{p} \in \mathcal{W}$, *the* 1*-forms* $dx_1(\mathbf{p}), \ldots, dx_n(\mathbf{p})$ *form a basis of* $\mathcal{M}_{\mathbf{p}}^*$. *Hence*

$$\{\, \omega_{\mathbf{p}} \mid \omega \in \mathfrak{X}(\mathcal{M})^* \,\} = \mathcal{M}_{\mathbf{p}}^*.$$

(ii) *The set* $\{dx_1, \ldots, dx_n\}$ *is a basis for* $\mathfrak{X}(\mathcal{W})^*$. *Thus for any* $\omega \in \mathfrak{X}(\mathcal{W})^*$, *we can write*

$$\omega = \sum_{i=1}^{n} \omega\left(\frac{\partial}{\partial x_i}\right) dx_i.$$

24.7 Exercises

1. If $\mathcal{M}_1$ and $\mathcal{M}_2$ are differentiable manifolds, show that $\mathcal{M}_1 \times \mathcal{M}_2$ is a differentiable manifold.

2. Show that each tangent space $\mathcal{M}_{\mathbf{p}}$ to a differentiable manifold $\mathcal{M}$ is itself a differentiable manifold.

3. Let $\Phi\colon \mathbb{R} \to \mathbb{R}$ be defined by $\Phi(t) = t^3$. Show that the complete atlas $\widetilde{\mathfrak{A}}_2$ on $\mathbb{R}$ containing the patch $(\Phi, \mathbb{R})$ is different from the usual complete atlas $\widetilde{\mathfrak{A}}_1$ containing the identity patch $(\mathbf{1}, \mathbb{R})$. Show that $(\mathbb{R}, \widetilde{\mathfrak{A}}_1)$ and $(\mathbb{R}, \widetilde{\mathfrak{A}}_2)$ are nonetheless diffeomorphic.

4. Recall the definition of real projective space $\mathbb{RP}^n$ in Exercise 12 of the previous chapter, and define again $p\colon S^n(1) \mapsto \mathbb{RP}^n$ by $p(\mathbf{a}) = \{\mathbf{a}, -\mathbf{a}\}$. Let

$$\mathcal{P}_j = \{\, p(a_1, \ldots, a_n) \mid a_j \neq 0\}.$$

Show that:

(a) Each $\mathcal{P}_j$ is an open subset of $\mathbb{RP}^n$.

(b) $\mathbb{RP}^n$ is the union of all the $\mathcal{P}_j$'s.

(c) For each j, there is a homeomorphism $\psi_j\colon \mathcal{P}_j \to \mathbb{R}^n$.

(d) $\mathbb{RP}^n$ is a compact differentiable manifold of dimension n.

5. On $\mathbb{R}^3$ consider the vector fields

$$\mathbf{X} = x^2\frac{\partial}{\partial x} + y\frac{\partial}{\partial z} \qquad \text{and} \qquad \mathbf{Y} = y^3\frac{\partial}{\partial y},$$

and the function $f\colon \mathbb{R}^3 \to \mathbb{R}$ defined by $f(p_1, p_2, p_3) = {p_1}^2 p_2 p_3$. Compute

(a) $[\mathbf{X}, \mathbf{Y}]_{(1,0,1)}$

(b) $(f\,\mathbf{Y})_{(1,0,1)}$

(c) $(\mathbf{Y}f)(1, 0, 1)$

(d) $f_{*(1,0,1)}\big(\mathbf{Y}_{(1,0,1)}\big)$.

Give examples of vector fields $\mathbf{X}$ and $\mathbf{Y}$ on $\mathbb{R}^2$ for which

$$\mathbf{X}_{(0,0)} = \mathbf{Y}_{(0,0)} \qquad \text{but} \qquad [\mathbf{X}, \mathbf{Y}]_{(0,0)} \neq \mathbf{0}.$$

6. Let $\Phi\colon \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $\Phi(p_1, p_2, p_3) = (p_1, p_2)$, and let $\mathbf{Z}$ be the vector field on $\mathbb{R}^3$ given by

$$\mathbf{Z} = \cos z\frac{\partial}{\partial x} + \sin z\frac{\partial}{\partial y} + \frac{\partial}{\partial z},$$

where x, y, z are the natural coordinate functions of $\mathbb{R}^3$. Show that there is no vector field $\mathbf{Y}$ on $\mathbb{R}^2$ for which

$$\Phi_{*\mathbf{p}}(\mathbf{Z}) = \mathbf{Y}_{\Phi(\mathbf{p})}$$

for all $\mathbf{p} \in \mathbb{R}^3$.

7. Set $\mathcal{W} = \{(u, v) \in \mathbb{R}^2 \mid uv > 0\}$. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ and $g\colon \mathcal{W} \to \mathbb{R}^3$ be defined by

$$f(u, v) = (u^2 \sin v,\ e^{uv}) \quad \text{and} \quad g(u, v) = (uv^2,\ \log(uv),\ v \sin u).$$

Compute the matrices of f_* and g_* with respect to the standard bases of vector fields on $\mathbb{R}^2$ and $\mathbb{R}^3$.

8. Prove Lemma 24.62.

9. Define, for each patch $(\mathbf{x},\mathcal{U})$ on an n-dimensional differentiable manifold $\mathcal{M}$, an associated patch $\widetilde{\mathbf{x}}\colon \mathcal{U}\times\mathbb{R}^n \to T(\mathcal{M})$ on the tangent bundle of $\mathcal{M}$ by setting

$$\widetilde{\mathbf{x}}(\mathbf{p},\mathbf{q}) = \left(\mathbf{x}(\mathbf{p}),\ \mathbf{x}_{*\mathbf{p}}\left(\sum_{i=1}^{n} q_i \frac{\partial}{\partial u_i}\right)\right),$$

where $\mathbf{p}\in\mathcal{U}$ and $\mathbf{q}=(q_1,\dots,q_n)$. Show that the tangent bundle $T(\mathcal{M})$ becomes a differentiable manifold with atlas

$$\left\{\,(\widetilde{\mathbf{x}},\,\mathcal{U}\times\mathbb{R}^n)\;\middle|\;(\mathbf{x},\mathcal{U}) \text{ is a patch on } \mathcal{M}\,\right\}.$$

10. Define, for fixed $a>0$, a mapping from an open set of $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$ by

$$\mathsf{stereo}(q_0,q_1,\dots,q_n) = \frac{a}{a-q_0}(q_1,\dots,q_n). \tag{24.20}$$

Referring to page 813, show that the composition $\mathsf{stereo}\circ\mathsf{north}$ equals the identity map from $\mathbb{R}^n$ to itself. Observe that (24.20) generalizes the mapping of the same name on page 730 to the case of arbitary n and a.