Chapter 4

New Curves from Old

Any plane curve gives rise to other plane curves through a variety of general constructions. Each such construction can be thought of as a *function* which assigns one curve to another, and we shall discover some new curves in this manner. Four classic examples of constructing one plane curve from another are studied in the present chapter: *evolutes* in Sections 4.1 and 4.2, *involutes* in Section 4.3, *parallel curves* in Section 4.5 and *pedal curves* in Section 4.6.

Along the way, we show how to construct normal and tangent lines to a curve, and *osculating circles* to curves. We explain in Section 4.4 that the circle through three points on a plane curve tends to the osculating circle as the three points become closer and coincide. For the same reason, the evolute of a plane curve can be visualized by plotting a sufficient number of normal lines to the curve, as illustrated by the well-known design in Figure 4.12 on page 111.

4.1 Evolutes

A point $\mathbf{p} \in \mathbb{R}^2$ is called a *center of curvature* at $\mathbf{q}$ of a curve $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^2$, provided that there is a circle $\boldsymbol{\gamma}$ with center $\mathbf{p}$ which is tangent to $\boldsymbol{\alpha}$ at $\mathbf{q}$ such that the curvatures of the curves $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$, suitably oriented, are the same at $\mathbf{q}$. We shall see that this implies that there is a line ℓ from $\mathbf{p}$ to $\boldsymbol{\alpha}$ which meets $\boldsymbol{\alpha}$ perpendicularly at $\mathbf{q}$, and the distance from $\mathbf{p}$ to $\mathbf{q}$ is the radius of curvature of $\boldsymbol{\alpha}$ at $\mathbf{q}$, as defined on page 14. An example is shown in Figure 4.2.

The centers of curvature form a new plane curve, called the evolute of $\boldsymbol{\alpha}$, whose precise definition is as follows.

Definition 4.1. *The* ***evolute*** *of a regular plane curve* $\boldsymbol{\alpha}$ *is the curve given by*

$$\mathsf{evolute}[\boldsymbol{\alpha}](t) = \boldsymbol{\alpha}(t) + \frac{1}{\boldsymbol{\kappa 2}[\boldsymbol{\alpha}](t)} \frac{J\boldsymbol{\alpha}'(t)}{\|\boldsymbol{\alpha}'(t)\|}. \tag{4.1}$$

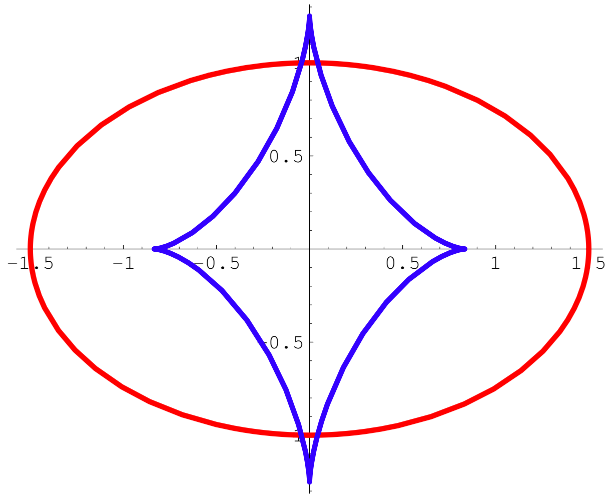

Figure 4.1: The ellipse and its evolute

It turns out that a circle with center $\mathsf{evolute}[\boldsymbol{\alpha}](t)$ and radius $1/|\boldsymbol{\kappa 2}[\boldsymbol{\alpha}](t)|$ will be tangent to the plane curve $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(t)$. This is the circle, called the *osculating circle*, that best approximates $\boldsymbol{\alpha}$ near $\boldsymbol{\alpha}(t)$; it is studied in Section 4.4.

Using Formula (1.12), page 14, we see that the formula for the evolute can be written more succinctly as

$$\mathsf{evolute}[\boldsymbol{\alpha}] = \boldsymbol{\alpha} + \frac{\|\boldsymbol{\alpha}'\|^2}{\boldsymbol{\alpha}'' \cdot J(\boldsymbol{\alpha}')} J\boldsymbol{\alpha}'. \tag{4.2}$$

An easy consequence of (4.1) and (1.15) is the following important fact.

Lemma 4.2. *The definition of evolute of a curve $\boldsymbol{\alpha}$ is independent of parametrization, so that*

$$\mathsf{evolute}[\boldsymbol{\alpha} \circ h] = \mathsf{evolute}[\boldsymbol{\alpha}] \circ h,$$

for any differentiable function $h\colon (c,d) \to (a,b)$.

The evolute of a circle consists of a single point. The evolute of any plane curve $\boldsymbol{\gamma}$ can be described physically. Imagine light rays starting at all points of the trace of $\boldsymbol{\gamma}$ and propagating down the normals of $\boldsymbol{\gamma}$. In the case of a circle, these rays focus perfectly at the center, so for $\boldsymbol{\gamma}$ the focusing occurs along the centers of best fitting circles, that is, along the evolute of $\boldsymbol{\gamma}$.

A more interesting example is the evolute of the ellipse $x^2/a^2 + y^2/b^2 = 1$, parametrized on page 21. This evolute is the curve $\boldsymbol{\gamma}$ defined by

$$\boldsymbol{\gamma}(t) = \left(\frac{(a^2 - b^2)\cos^3 t}{a}, \; \frac{(a^2 - b^2)\sin^3 t}{b} \right), \tag{4.3}$$

and is an astroid[1] (see Exercise 1 of Chapter 1). Of course, setting $a = b$ confirms that the evolute of a circle is simply its center-point. Figure 4.1 represents an ellipse and its evolute simultaneously, though the focussing property is best appreciated by viewing Figure 4.12.

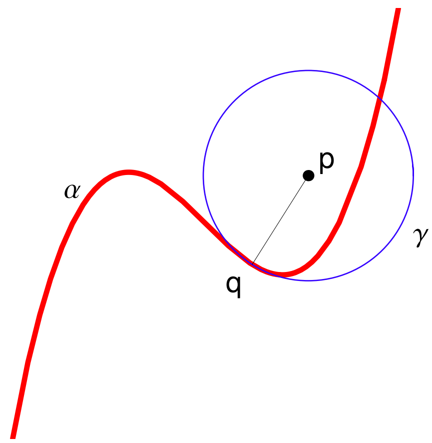

Figure 4.2: A center of curvature on a cubic curve

The notions of tangent line and normal line to a curve are clear intuitively; here is the mathematical definition.

Definition 4.3. *The **tangent line** and **normal line** to a curve $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^2$ at $\boldsymbol{\alpha}(t)$ are the straight lines passing through $\boldsymbol{\alpha}(t)$ with velocity vectors equal to $\boldsymbol{\alpha}'(t)$ and $J\boldsymbol{\alpha}'(t)$, respectively.*

Next, we obtain a characterization of the evolute of a curve in terms of tangent lines and normal lines, and also determine the singular points of the evolute.

Theorem 4.4. *Let $\boldsymbol{\beta}\colon (a,b) \to \mathbb{R}^2$ be a unit-speed curve. Then*

(i) *the evolute of $\boldsymbol{\beta}$ is the unique curve of the form $\boldsymbol{\gamma} = \boldsymbol{\beta} + fJ\boldsymbol{\beta}'$ for some function f for which the tangent line to $\boldsymbol{\gamma}$ at each point $\boldsymbol{\gamma}(s)$ coincides with the normal line to $\boldsymbol{\beta}$ at $\boldsymbol{\beta}(s)$.*

(ii) *Suppose that $\boldsymbol{\kappa 2}[\boldsymbol{\beta}]$ is nowhere zero. The singular points of the evolute of $\boldsymbol{\beta}$ occur at those values of s for which $\boldsymbol{\kappa 2}[\boldsymbol{\beta}]'(s) = 0$.*

Proof. When we differentiate (4.1) and use Lemma 1.21, page 16, we obtain

$$\mathsf{evolute}[\boldsymbol{\beta}]' = -\frac{\boldsymbol{\kappa 2}[\boldsymbol{\beta}]'}{\big(\boldsymbol{\kappa 2}[\boldsymbol{\beta}]\big)^2} J\boldsymbol{\beta}'. \tag{4.4}$$

[1]Note that an *astroid* is a four-cusped curve, but that an *asteroid* is a small planet. This difference persists in English, French, Spanish and Portugese, but curiously in Italian the word for both notions is *asteroide*.

Hence the tangent line to evolute$[\boldsymbol{\beta}]$ at evolute$[\boldsymbol{\beta}](s)$ coincides with the normal line to $\boldsymbol{\beta}$ at $\boldsymbol{\beta}(s)$.

Conversely, suppose $\boldsymbol{\gamma} = \boldsymbol{\beta} + f J\boldsymbol{\beta}'$. Again, using Lemma 1.21, we compute

$$\boldsymbol{\gamma}' = \big(1 - f\boldsymbol{\kappa2}[\boldsymbol{\beta}]\big)\boldsymbol{\beta}' + f' J\boldsymbol{\beta}'.$$

If the tangent line to $\boldsymbol{\gamma}$ at each point $\boldsymbol{\gamma}(s)$ coincides with the normal line to $\boldsymbol{\beta}$ at $\boldsymbol{\beta}(s)$, then $f = 1/\boldsymbol{\kappa2}[\boldsymbol{\beta}]$, and so $\boldsymbol{\gamma}$ is the evolute of $\boldsymbol{\beta}$.

This proves (i); then (ii) is a consequence of (4.4). ∎

For the parametrization ellipse$[a,b]$, it is verified in Notebook 4 that

$$\boldsymbol{\kappa2}'(t) = \frac{3ab(b^2 - a^2)\sin 2t}{2(a^2\sin^2 t + b^2\cos^2 t)^{5/2}}.$$

It therefore follows that the evolute of an ellipse is singular when t assumes one of the four values 0, $\pi/2$, π, $3\pi/2$. This is confirmed by both differentiating (4.3) and inspecting the plot in Figure 4.1.

4.2 Iterated Evolutes

An obvious problem is to see what happens when one applies the evolute construction repeatedly. Computations are carried out in Notebook 4 to find the evolute of an evolute of a curve, and to iterate the procedure.

As an example of this phenomenon, Figure 4.3 plots the first three evolutes of the cissoid, parametrized on page 48. In spite of the fact that the cissoid has a cusp, its first evolute (passing through $(-2,0)$) and the third evolute (passing through $(0,0)$) do not.

Another example of a curve with a cusp whose evolute has no cusp is the tractrix, parametrized on page 51. It turns out that the evolute of a tractrix is a catenary. By direct computation, the evolute of tractrix$[a]$ is the curve

$$t \mapsto a\left(\frac{1}{\sin t},\ \log\Big(\tan\frac{t}{2}\Big)\right).$$

To see that this curve is actually a catenary, let $\tau = \tan(t/2)$ and $u = \log\tau$. Then

$$e^u + e^{-u} = \frac{\tau^2+1}{\tau} = \frac{2}{\sin t}.$$

Thus the evolute of the tractrix can be reparametrized as $u \mapsto a(\cosh u,\, u)$, which is a multiple of catenary$[1]$ as defined on page 47.

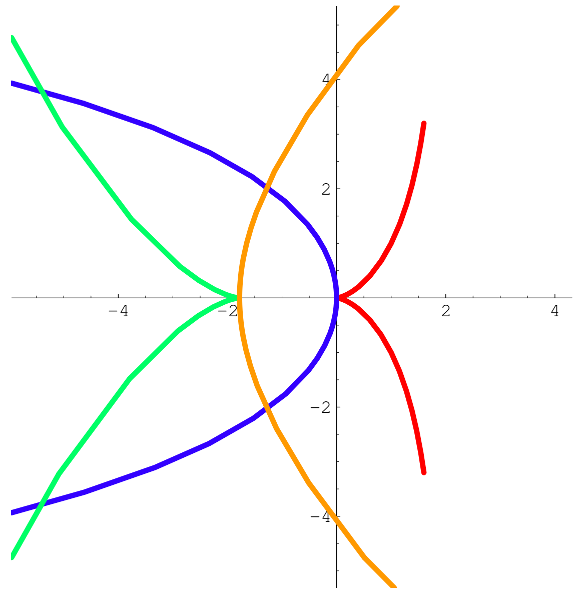

Figure 4.3: Iterated evolutes of cissoid[1]

For Figure 4.4, the evolute of the tractrix was computed and plotted automatically; it is the smooth curve that passes through the cusp. A catenary was then added to the picture off-center, but close enough to emphasize that it is the same curve as the evolute.

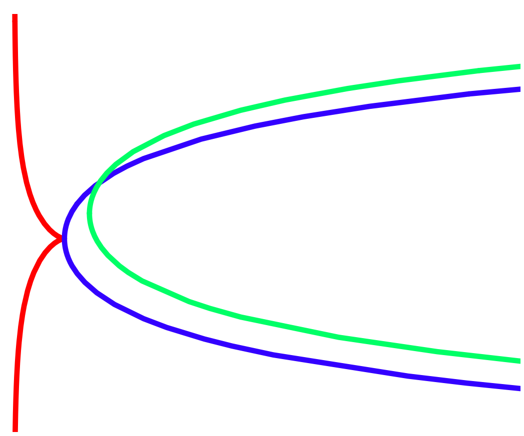

Figure 4.4: tractrix[1] and its catenary evolute

4.3 Involutes

The involute is a geometrically important operation that is inverse to the map $\boldsymbol{\alpha} \mapsto \mathsf{evolute}[\boldsymbol{\alpha}]$ that associates to a curve its evolute. In fact, the evolute is related to the involute in the same way that differentiation is related to indefinite integration. Just like the latter, the operation of taking the involute depends on an arbitrary constant. Furthermore, we shall prove (Theorem 4.9) that the evolute of the involute of a curve $\boldsymbol{\gamma}$ is again $\boldsymbol{\gamma}$; this corresponds to the fact that the derivative of the indefinite integral of a function f is again f.

We first give the definition of the involute of a unit-speed curve.

Definition 4.5. *Let $\boldsymbol{\beta}\colon (a,b) \to \mathbb{R}^2$ be a unit-speed curve, and let $a < c < b$. The* ***involute*** *of $\boldsymbol{\beta}$ starting at $\boldsymbol{\beta}(c)$ is the curve given by*

$$\mathsf{involute}[\boldsymbol{\beta}, c](s) = \boldsymbol{\beta}(s) + (c - s)\boldsymbol{\beta}'(s). \tag{4.5}$$

Whereas the evolute of a plane curve $\boldsymbol{\beta}$ is a linear combination of $\boldsymbol{\beta}$ and $J\boldsymbol{\beta}'$, an involute of $\boldsymbol{\beta}$ is a linear combination of $\boldsymbol{\beta}$ and $\boldsymbol{\beta}'$. Note that although we use s as the arc length parameter of $\boldsymbol{\beta}$, it is *not* necessarily an arc length parameter for the involute of $\boldsymbol{\beta}$.

The formula for the involute of an arbitrary-speed curve needs the arc length function defined on page 12.

Lemma 4.6. *Let $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^2$ be a regular arbitrary-speed curve. Then the involute of $\boldsymbol{\alpha}$ starting at c (where $a < c < b$) is given by*

$$\mathsf{involute}[\boldsymbol{\alpha}, c](t) = \boldsymbol{\alpha}(t) + \big(s_\alpha(c) - s_\alpha(t)\big)\frac{\boldsymbol{\alpha}'(t)}{\big\|\boldsymbol{\alpha}'(t)\big\|}, \tag{4.6}$$

where $t \mapsto s_\alpha(t)$ denotes the arc length of $\boldsymbol{\alpha}$ measured from an arbitary point.

The involute of a curve can be described geometrically.

Theorem 4.7. *An involute of a regular plane curve $\boldsymbol{\beta}$ is formed by unwinding a taut string which has been wrapped around $\boldsymbol{\beta}$.*

This result is illustrated in Figure 4.5, in which the string has been 'cut' at the point $\boldsymbol{\beta}(c)$ on the curve, and gradually unwound from that point.

Proof. Without loss of generality, we may suppose that $\boldsymbol{\beta}$ has unit-speed. Then

$$\mathsf{involute}[\boldsymbol{\beta}, c](s) - \boldsymbol{\beta}(s) = (c - s)\boldsymbol{\beta}'(s),$$

so that

$$\big\|\mathsf{involute}[\boldsymbol{\beta}, c](s) - \boldsymbol{\beta}(s)\big\| = |s - c|. \tag{4.7}$$

Here $|s - c|$ is the distance from $\boldsymbol{\beta}(s)$ to $\boldsymbol{\beta}(c)$ measured along the curve $\boldsymbol{\beta}$, while the left-hand side of (4.7) is the distance from $\mathsf{involute}[\boldsymbol{\beta}, c](s)$ to $\boldsymbol{\beta}(s)$ measured along the tangent line to $\boldsymbol{\beta}$ emanating from $\boldsymbol{\beta}(s)$. ∎

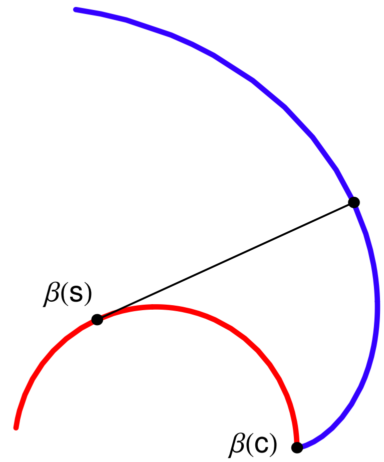

Figure 4.5: Definition of an involute

The most famous involute is that of a circle. With reference to (1.25),

$$\text{involute}[\text{circle}[a], b](t) = a\big(\cos t + (-b + t)\sin t,\ (b - t)\cos t + \sin t\big).$$

An example is visible in Figure 4.7 on page 107, in which the operation of taking the involute has been iterated. (This is carried out analytically in Exercise 6).

The involute of the figure eight (2.4) requires a complicated integral, which is computed numerically in Notebook 4 to obtain Figure 4.6 and a selection of normal lines.

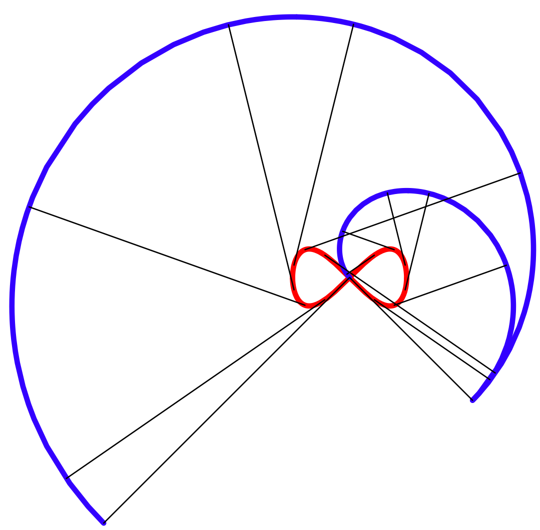

Figure 4.6: Involute of a figure eight

Next, we find a useful relation between the curvature of a curve and that of its involute.

Lemma 4.8. *Let $\boldsymbol{\beta}\colon (a,b) \to \mathbb{R}^2$ be a unit-speed curve, and let $\boldsymbol{\gamma}$ be the involute of $\boldsymbol{\beta}$ starting at c, where $a < c < b$. Then the curvature of $\boldsymbol{\gamma}$ is given by*

$$\boldsymbol{\kappa 2}[\gamma](s) = \frac{\operatorname{sign}\big(\boldsymbol{\kappa 2}[\boldsymbol{\beta}](s)\big)}{|s-c|}. \tag{4.8}$$

Proof. First, we use (4.5) and Lemma 1.21 to compute

$$\boldsymbol{\gamma}'(s) = (c-s)\boldsymbol{\beta}''(s) = (c-s)\boldsymbol{\kappa 2}[\boldsymbol{\beta}](s)\, J\boldsymbol{\beta}'(s), \tag{4.9}$$

and

$$\begin{aligned} \boldsymbol{\gamma}''(s) &= -\boldsymbol{\kappa 2}[\boldsymbol{\beta}](s)\, J\boldsymbol{\beta}'(s) + (c-s)\boldsymbol{\kappa 2}[\boldsymbol{\beta}]'(s)\, J\boldsymbol{\beta}'(s) \\ &\qquad + (c-s)\boldsymbol{\kappa 2}[\boldsymbol{\beta}](s)\, J\boldsymbol{\beta}''(s) \\ &= \Big(-\boldsymbol{\kappa 2}[\boldsymbol{\beta}](s) + (c-s)\boldsymbol{\kappa 2}[\boldsymbol{\beta}]'(s)\Big) J\boldsymbol{\beta}'(s) \\ &\qquad - (c-s)\big(\boldsymbol{\kappa 2}[\boldsymbol{\beta}](s)\big)^2 \boldsymbol{\beta}'(s). \end{aligned} \tag{4.10}$$

From (4.9) and (4.10), we get

$$\boldsymbol{\gamma}''(s) \cdot J\boldsymbol{\gamma}'(s) = (c-s)^2\big(\boldsymbol{\kappa 2}[\boldsymbol{\beta}](s)\big)^3. \tag{4.11}$$

Now (4.8) follows from (4.9), (4.11) and the definition of $\boldsymbol{\kappa 2}[\gamma]$. ∎

Lemma 4.8 implies that the absolute value of the curvature of the involute of a curve is always decreasing as a function of s in the range $s \geqslant c$. This can be seen clearly in Figures 4.6 and 4.7.

Theorem 4.9. *Let $\boldsymbol{\beta}\colon (a,b) \to \mathbb{R}^2$ be a unit-speed curve and let $\boldsymbol{\gamma}$ be the involute of $\boldsymbol{\beta}$ starting at c, where $a < c < b$. Then the evolute of $\boldsymbol{\gamma}$ is $\boldsymbol{\beta}$.*

Proof. By definition the evolute of $\boldsymbol{\gamma}$ is the curve $\boldsymbol{\zeta}$ given by

$$\boldsymbol{\zeta}(s) = \boldsymbol{\gamma}(s) + \frac{1}{\boldsymbol{\kappa 2}[\gamma](s)} \frac{J\boldsymbol{\gamma}'(s)}{\|\boldsymbol{\gamma}'(s)\|}. \tag{4.12}$$

When we substitute (4.5), (4.8) and (4.9) into (4.12), we get

$$\begin{aligned} \boldsymbol{\zeta}(s) &= \boldsymbol{\beta}(s) + (c-s)\boldsymbol{\beta}'(s) + \frac{|c-s|}{\operatorname{sign}\big(\boldsymbol{\kappa 2}[\boldsymbol{\beta}](s)\big)} \frac{(c-s)\boldsymbol{\kappa 2}[\boldsymbol{\beta}](s)\, J^2\boldsymbol{\beta}'(s)}{\big\|(c-s)\boldsymbol{\kappa 2}[\boldsymbol{\beta}](s)\, J\boldsymbol{\beta}'(s)\big\|} \\ &= \boldsymbol{\beta}(s). \end{aligned}$$

Thus $\boldsymbol{\beta}$ and $\boldsymbol{\zeta}$ coincide. ∎

The previous result is consistent with thinking of ‘evolution’ as differentiation, and ‘involution’ as integration, as explained at the start of this section. That being the case, one would expect a sequel to Theorem 4.9 to assert that the involute of the evolute of a curve $\boldsymbol{\beta}$ is the same as $\boldsymbol{\beta}$ ‘up to a constant’. The appropriate notion is contained in Definition 4.13 below: it can be shown that the involute of the evolute of $\boldsymbol{\beta}$ is actually a *parallel curve* to $\boldsymbol{\beta}$. Examples are given in Notebook 4.

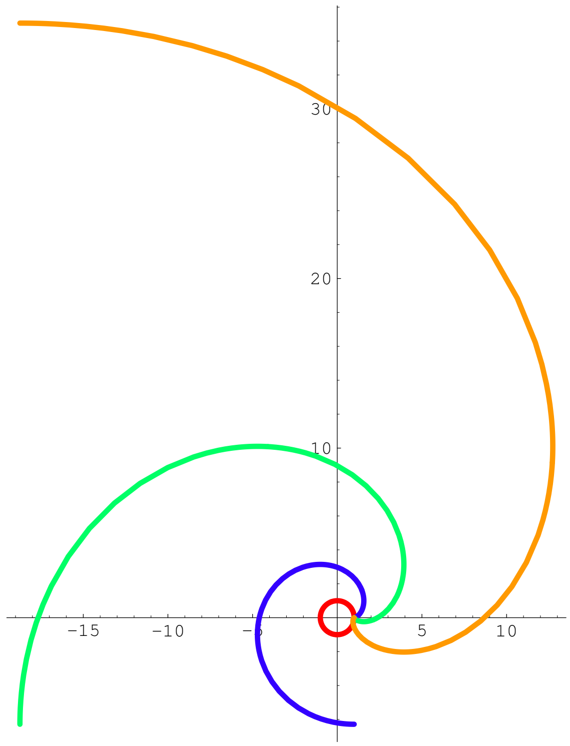

Figure 4.7: A circle and three successive involutes

4.4 Osculating Circles to Plane Curves

Just as the tangent is the best *line* that approximates a curve at one of its points $\mathbf{p}$, the osculating circle is the best *circle* that approximates the curve at $\mathbf{p}$.

Definition 4.10. *Let $\boldsymbol{\alpha}$ be a regular plane curve defined on an interval (a,b), and let $a<t<b$ be such that $\boldsymbol{\kappa 2}[\boldsymbol{\alpha}](t)\neq 0$. Then the* ***osculating circle*** *to $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(t)$ is the circle of radius $1/|\boldsymbol{\kappa 2}[\boldsymbol{\alpha}](t)|$ and center*

$$\boldsymbol{\alpha}(t)+\frac{1}{\boldsymbol{\kappa 2}[\boldsymbol{\alpha}](t)}\frac{J\boldsymbol{\alpha}'(t)}{\|\boldsymbol{\alpha}'(t)\|}.$$

The dictionary definition of 'osculating' is kissing. In fact, the osculating circle at a point $\mathbf{p}$ on a curve approximates the curve much more closely than the tangent line. Not only do $\boldsymbol{\alpha}$ and its osculating circle at $\boldsymbol{\alpha}(t)$ have the same tangent line and normal line, but also the same curvature.

It is easy to see from equation (4.1) that

Lemma 4.11. *The centers of the osculating circles to a curve form the evolute to the curve.*

The osculating circles to a logarithmic spiral are a good example of close approximation to the curve. Figure 4.8 represents these circles without the spiral itself.

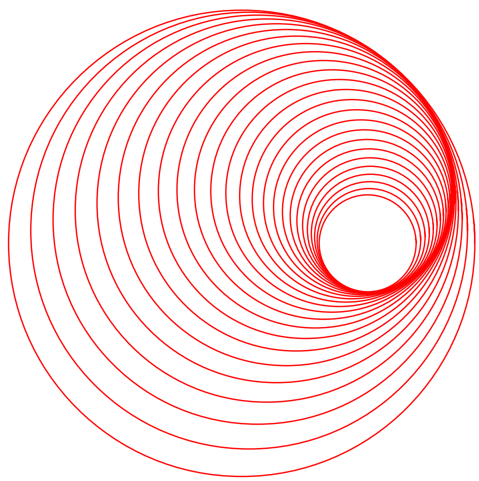

Figure 4.8: Osculating circles to logspiral$[1, -1.5]$

Next, we show that an osculating circle to a plane curve is the limit of circles passing through three points of the curve as the points tend to the point of contact of the osculating circle.

Theorem 4.12. *Let $\boldsymbol{\alpha}$ be a plane curve defined on an interval (a, b), and let $a < t_1 < t_2 < t_3 < b$. Denote by $\mathscr{C}(t_1, t_2, t_3)$ the circle passing through the points $\boldsymbol{\alpha}(t_1), \boldsymbol{\alpha}(t_2), \boldsymbol{\alpha}(t_3)$ provided these points are distinct and do not lie on the same straight line. Assume that $\boldsymbol{\kappa}\mathbf{2}[\boldsymbol{\alpha}](t_0) \neq 0$. Then the osculating circle to $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(t_0)$ is the circle*

$$\mathscr{C} = \lim_{\substack{t_1 \to t_0 \\ t_2 \to t_0 \\ t_3 \to t_0}} \mathscr{C}(t_1, t_2, t_3).$$

Proof. Denote by $\mathbf{p}(t_1, t_2, t_3)$ the center of $\mathscr{C}(t_1, t_2, t_3)$, and define $f\colon (a,b) \to \mathbb{R}$ by

$$f(t) = \big\|\boldsymbol{\alpha}(t) - \mathbf{p}(t_1, t_2, t_3)\big\|^2.$$

Then

$$\left\{ \begin{array}{rcl} f'(t) &=& 2\boldsymbol{\alpha}'(t) \cdot \big(\boldsymbol{\alpha}(t) - \mathbf{p}(t_1,t_2,t_3)\big), \\ f''(t) &=& 2\boldsymbol{\alpha}''(t) \cdot \big(\boldsymbol{\alpha}(t) - \mathbf{p}(t_1,t_2,t_3)\big) + 2\big\|\boldsymbol{\alpha}'(t)\big\|^2. \end{array} \right. \tag{4.13}$$

Since f is differentiable and $f(t_1) = f(t_2) = f(t_3)$, there exist u_1 and u_2 with $t_1 < u_1 < t_2 < u_2 < t_3$ such that

$$f'(u_1) = f'(u_2) = 0. \tag{4.14}$$

Similarly, there exists v with $u_1 < v < u_2$ such that

$$f''(v) = 0. \tag{4.15}$$

(Equations (4.14) and (4.15) follow from Rolle's theorem[2]. See for example, [Buck, page 90].) Clearly, as t_1, t_2, t_3 tend to t_0, so do u_1, u_2, v. Equations (4.13)–(4.15) imply that

$$\left\{ \begin{array}{rcl} \boldsymbol{\alpha}'(t_0) \cdot (\boldsymbol{\alpha}(t_0) - \mathbf{p}) &=& 0, \\ \boldsymbol{\alpha}''(t_0) \cdot (\boldsymbol{\alpha}(t_0) - \mathbf{p}) &=& -\big\|\boldsymbol{\alpha}'(t_0)\big\|^2, \end{array} \right. \tag{4.16}$$

where

$$\mathbf{p} = \lim_{\substack{t_1 \to t_0 \\ t_2 \to t_0 \\ t_3 \to t_0}} \mathbf{p}(t_1, t_2, t_3).$$

It follows from (4.16) and the definition of $\boldsymbol{\kappa 2}$ that

$$\boldsymbol{\alpha}(t_0) - \mathbf{p} = \frac{-1}{\boldsymbol{\kappa 2}[\boldsymbol{\alpha}](t_0)} \frac{J\boldsymbol{\alpha}'(t_0)}{\big\|\boldsymbol{\alpha}'(t_0)\big\|}.$$

Thus, by definition, $\mathscr{C}$ is the osculating circle to $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(t)$. ∎

Figure 4.9 plots various circles, each passing through three points on the parabola $4y = x^2$. As the three points converge to the vertex of the parabola, the circle through the three points converges to the osculating circle at the vertex. Since the curvature of the parabola at the vertex is $1/2$, the osculating circle at the vertex has radius exactly 2.

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[2]Michel Rolle (1652–1719). French mathematician, who resisted the infinitesimal techniques of calculus.

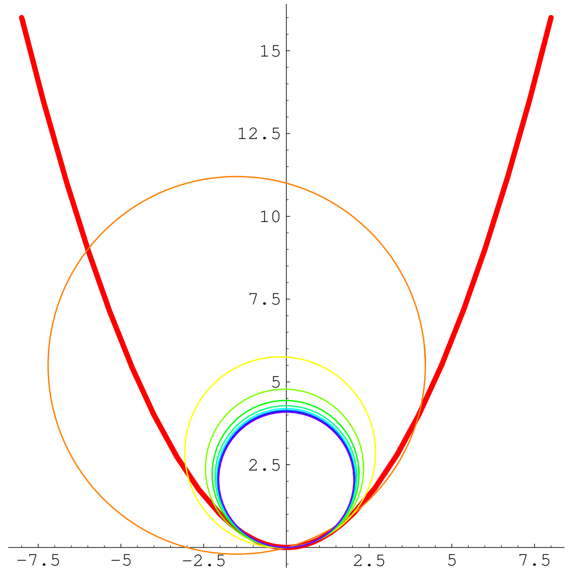

Figure 4.9: Circles converging to an osculating circle of a parabola

4.5 Parallel Curves

It is appropriate to begin this section by illustrating the concept of tangent and normal lines. The tangent line to a plane curve at a point **p** on the curve is the best linear approximation to the curve at **p**, and the normal line is the tangent line rotated by $\pi/2$.

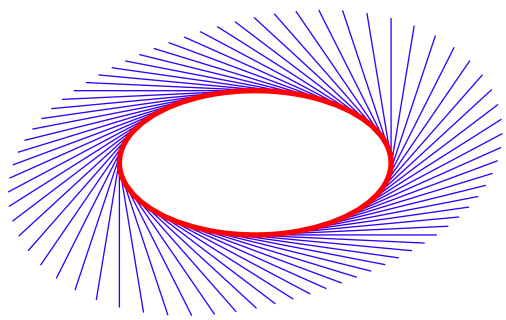

Figure 4.10: Tangent lines to ellipse$[\frac{3}{2}, 1]$

Having drawn some short tangent lines to an ellipse (parametrized on page 21), as if to give it fur, we draw normal lines to the same ellipse in Figure 4.11. Longer normal lines may intersect one another, as we see from Figure 4.12, in which we can clearly distinguish the evolute of the ellipse.

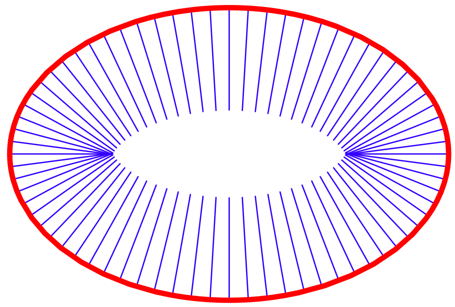

Figure 4.11: Normal lines to ellipse$[\frac{3}{2}, 1]$

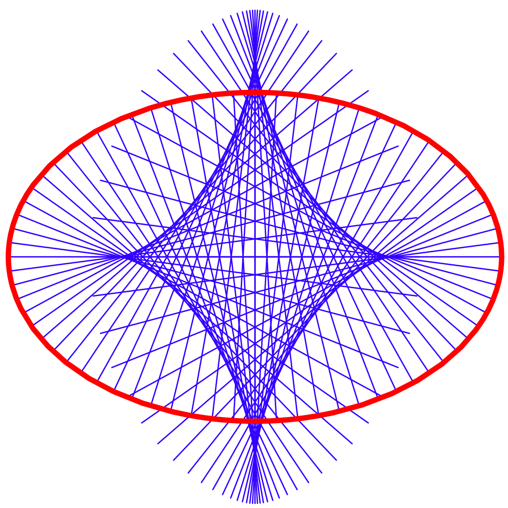

Figure 4.12: Intersecting normals to the same ellipse

We shall now construct a curve $\boldsymbol{\gamma}$ at a fixed distance $r > 0$ from a given curve $\boldsymbol{\alpha}$, where r is not too large. Let $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ be defined on an interval (a, b); then we require

$$\|\boldsymbol{\gamma}(t) - \boldsymbol{\alpha}(t)\| = r \qquad \text{and} \qquad \big(\boldsymbol{\gamma}(t) - \boldsymbol{\alpha}(t)\big) \cdot \boldsymbol{\alpha}'(t) = 0,$$

for $a < t < b$. This leads us to the next definition.

Definition 4.13. *A **parallel curve** to a regular plane curve $\boldsymbol{\alpha}$ at a distance r is the plane curve given by*

$$\text{parcurve}[\boldsymbol{\alpha}, r](t) = \boldsymbol{\alpha}(t) + \frac{r\, J\boldsymbol{\alpha}'(t)}{\|\boldsymbol{\alpha}'(t)\|}. \tag{4.17}$$

Actually, we can now allow r in (4.17) to be either positive or negative, in order to obtain parallel curves on either side of $\boldsymbol{\alpha}$, without changing t. The definition of parallel curve does not depend on the choice of positive reparametrization. In fact, it is not hard to prove

Lemma 4.14. *Let* $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^2$ *be a plane curve, and let* $h\colon (c,d) \to (a,b)$ *be differentiable. Then*

$$\mathsf{parcurve}[\boldsymbol{\alpha} \circ h,\ r](u) = \mathsf{parcurve}[\boldsymbol{\alpha},\ r \operatorname{sign} h'](h(u)).$$

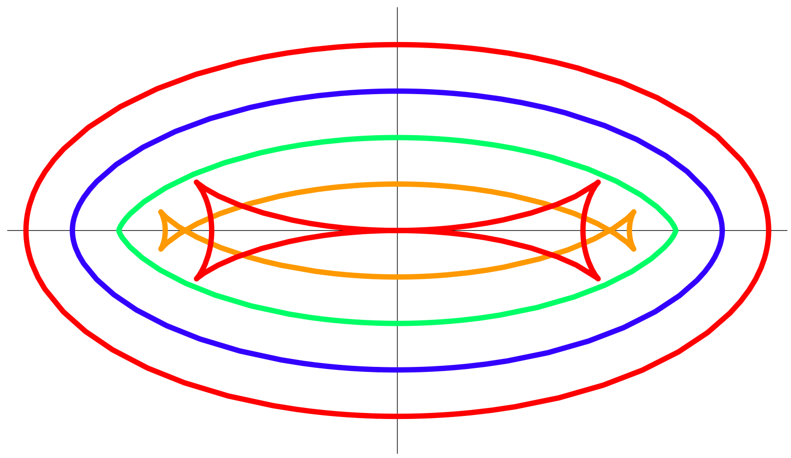

Figure 4.13: Four parallel curves to $\mathsf{ellipse}[2,1]$

Figure 4.13 illustrates some parallel curves to an ellipse. It shows that if $|r|$ is too large, a parallel curve may intersect itself and $|r|$ will not necessarily represent distance to the original curve. We can estimate when this first happens, and at the same time compute the curvature of the parallel curve.

Lemma 4.15. *Let* $\boldsymbol{\alpha}$ *be a regular plane curve. Then the curve* $\mathsf{parcurve}[\boldsymbol{\alpha}, r]$ *is regular at those* t *for which* $1 - r\,\boldsymbol{\kappa 2}[\boldsymbol{\alpha}](t) \neq 0$. *Furthermore, its curvature is given by*

$$\boldsymbol{\kappa 2}[\mathsf{parcurve}[\boldsymbol{\alpha}, r]](t) = \frac{\boldsymbol{\kappa 2}[\boldsymbol{\alpha}](t)}{\big|1 - r\,\boldsymbol{\kappa 2}[\boldsymbol{\alpha}](t)\big|}.$$

Proof. By Theorem 1.20, page 16, and Lemma 4.14 we can assume that $\boldsymbol{\alpha}$ has unit speed. Write $\boldsymbol{\beta}(t) = \mathsf{parcurve}[\boldsymbol{\alpha}, r](t)$. Then $\boldsymbol{\beta} = \boldsymbol{\alpha} + r\,J\boldsymbol{\alpha}'$ so that by Lemma 1.21 we have

$$\boldsymbol{\beta}' = \boldsymbol{\alpha}' + r\,J\boldsymbol{\alpha}'' = \boldsymbol{\alpha}' + r\,J^2\boldsymbol{\kappa 2}[\boldsymbol{\alpha}]\boldsymbol{\alpha}' = \big(1 - r\,\boldsymbol{\kappa 2}[\boldsymbol{\alpha}]\big)\boldsymbol{\alpha}'.$$

The regularity statement follows. Also, we compute

$$\boldsymbol{\beta}'' = \big(1 - r\boldsymbol{\kappa 2}[\boldsymbol{\alpha}]\big)\boldsymbol{\kappa 2}[\boldsymbol{\alpha}]\,J\boldsymbol{\alpha}' - r\,\boldsymbol{\kappa 2}[\boldsymbol{\alpha}]'\boldsymbol{\alpha}'.$$

Hence

$$\boldsymbol{\kappa 2}[\boldsymbol{\beta}] = \frac{\boldsymbol{\beta}'' \cdot J\boldsymbol{\beta}'}{\|\boldsymbol{\beta}'\|^3} = \frac{\big(1 - r\,\boldsymbol{\kappa 2}[\boldsymbol{\alpha}]\big)^2 \boldsymbol{\kappa 2}[\boldsymbol{\alpha}]}{\big|1 - r\boldsymbol{\kappa 2}[\boldsymbol{\alpha}]\big|^3} = \frac{\boldsymbol{\kappa 2}[\boldsymbol{\alpha}]}{\big|1 - r\,\boldsymbol{\kappa 2}[\boldsymbol{\alpha}]\big|},$$

as stated. ▮

4.6 Pedal Curves

Let $\boldsymbol{\alpha}$ be a curve in the plane, and let $\mathbf{p} \in \mathbb{R}^2$. The locus of base points $\boldsymbol{\beta}(t)$ of a perpendicular line let down from $\mathbf{p}$ to the tangent line to $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(t)$ is called the pedal curve to $\boldsymbol{\alpha}$ with respect to $\mathbf{p}$. It follows that $\boldsymbol{\beta}(t) - \mathbf{p}$ is the projection of $\boldsymbol{\alpha}(t) - \mathbf{p}$ in the $J\boldsymbol{\alpha}'(t)$ direction, as shown in Figure 4.15. This enables us to give a more formal definition:

Definition 4.16. *The* ***pedal curve*** *of a regular curve* $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^2$ *with respect to a point* $\mathbf{p} \in \mathbb{R}^2$ *is defined by*

$$\mathsf{pedal}[\boldsymbol{\alpha}, \mathbf{p}](t) = \mathbf{p} + \frac{\big(\boldsymbol{\alpha}(t) - \mathbf{p}\big) \cdot J\boldsymbol{\alpha}'(t)}{\big\|\boldsymbol{\alpha}'(t)\big\|^2} J\boldsymbol{\alpha}'(t).$$

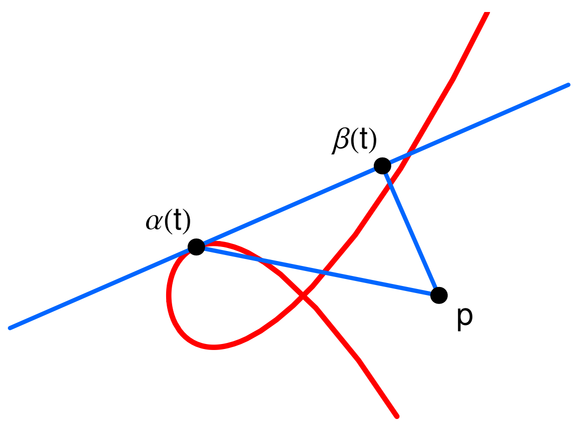

Figure 4.14: The definition of a pedal curve

The proof of the following lemma follows the model of Theorem 1.20; see Exercise 13.

Lemma 4.17. *The definition of the pedal curve of* $\boldsymbol{\alpha}$ *is independent of the parametrization of* $\boldsymbol{\alpha}$*, so that*

$$\mathsf{pedal}[\boldsymbol{\alpha} \circ h,\ \mathbf{p}] = \mathsf{pedal}[\boldsymbol{\alpha}, \mathbf{p}] \circ h.$$

Other examples are:

(i) The pedal curve of a parabola with respect to its vertex is a cissoid; see Exercise 12.

(ii) The pedal curve of a circle with respect to any point **p** other than the center of the circle is a limaçon. If **p** lies on the circumference of the circle, the limaçon reduces to a cardioid. The pedal curve of a circle with respect to its center is the circle itself; see Exercise 17.

(iii) The pedal curve of cardioid$[a]$ (see page 45) with respect to its cusp point is called ***Cayley's sextic***[3]. Its equation is

$$\mathsf{cayleysextic}[a](t) = 4a\left((2\cos t - 1)\cos^4\frac{t}{2},\ \sin\frac{3t}{2}\cos^3\frac{t}{2}\right).$$

Figure 4.16 plots the cardioid and Cayley's sextic simultaneously.

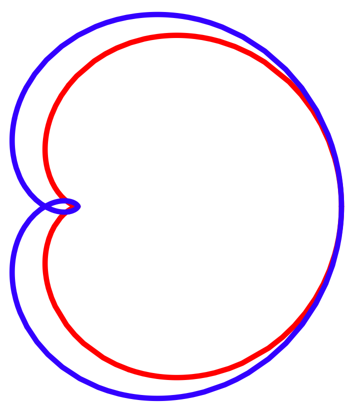

Figure 4.15: Cardioid and its pedal curve

(iv) The pedal curve of

$$\boldsymbol{\alpha}(t) = \left(t,\ \frac{t^3}{3} + \frac{1}{2}\right),$$

illustrated in Figure 4.14, has a singularity, discussed in [BGM, page 5].

3  Arthur Cayley (1821–1895). One of the leading English mathematicians of the 19$^{\text{th}}$ century; his complete works fill many volumes. Particularly known for his work on matrices, elliptic functions and nonassociative algebras. His first 14 professional years were spent as a lawyer; during that time he published over 250 papers. In 1863 Cayley was appointed Sadleirian professor of mathematics at Cambridge with a greatly reduced salary.

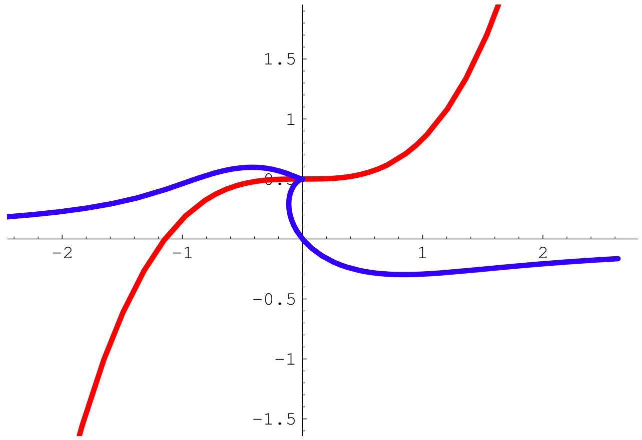

Figure 4.16: The cubic $y = \frac{1}{3}x^3 + \frac{1}{2}$ and its pedal curve

For more information about pedal curves see [Law, pages 46–49], [Lock, pages 153–160] and [Zwi, pages 150–158]. We shall return to pedal curves, and parametrize them with an angle, as part of the study of ovals in Section 6.8.

4.7 Exercises

M **1.** Plot evolutes of a cycloid, a cardioid and a logarithmic spiral. Show that the evolute of a cardioid is another cardioid, and that the evolute of a logarithmic spiral is another logarithmic spiral.

2. Determine conditions under which the evolute of a cycloid is another cycloid.

3. Prove Lemma 4.2.

M **4.** Plot normal lines to a cardioid, making the lines sufficiently long so that they intersect.

M **5.** Plot as one graph four parallel curves to a lemniscate. Do the same for a cardioid and a deltoid (see page 57).

6. Show that the curve defined by

$$\gamma(t) = a\,e^{it} \sum_{k=0}^{n} \frac{(-it)^k}{k!}$$

is the n^{th} involute starting at $(a, 0)$ of a circle of radius a.

7. A ***strophoid*** of a curve $\boldsymbol{\alpha}$ with respect to a point $\mathbf{p} \in \mathbb{R}^2$ is a curve $\boldsymbol{\gamma}$ such that

$$\big\|\boldsymbol{\alpha}(t) - \boldsymbol{\gamma}(t)\big\| = \big\|\boldsymbol{\alpha}(t) - \mathbf{p}\big\| \qquad \text{and} \qquad \boldsymbol{\gamma}(t) = s\,\boldsymbol{\alpha}(t),$$

for some s. Show that $\boldsymbol{\alpha}$ has two strophoids and find the equations for them. (A special case was defined in Exercise 1 on page 85.)

8. Show that the involute of a catenary is a tractrix.

M **9.** A ***point of inflection*** of a plane curve $\boldsymbol{\alpha}$ is defined to be a point $\boldsymbol{\alpha}(t_0)$ for which $\boldsymbol{\kappa}\mathbf{2}[\boldsymbol{\alpha}](t_0) = 0$; a ***strong inflection point*** is a point $\boldsymbol{\alpha}(t_0)$ for which there exists $\varepsilon > 0$ such that $\boldsymbol{\kappa}\mathbf{2}[\boldsymbol{\alpha}](t)$ is negative for $t_0 - \varepsilon < t < t_0$ and positive for $t_0 < t < t_0 + \varepsilon$, or vice versa.

(a) Show that if $\boldsymbol{\alpha}(t_0)$ is a strong inflection point, and if $t \mapsto \boldsymbol{\kappa}\mathbf{2}[\boldsymbol{\alpha}](t)$ is continuous at t_0, then $\boldsymbol{\alpha}(t_0)$ is an inflection point.

(b) For the curve $t \mapsto (t^3, t^5)$ show that $(0,0)$ is a strong inflection point which is not an inflection point. Plot the curve.

10. Let $\boldsymbol{\alpha}(t_0)$ be a strong inflection point of a curve $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^2$. Show that any involute of $\boldsymbol{\alpha}$ must have discontinuous curvature at t_0. This accounts for the cusps on the involutes of a figure eight and of a cubic parabola.

M **11.** Find and draw the pedal curve of (i) an ellipse with respect to its center, and (ii) a catenary with respect to the origin.

12. Find the parametric form of the pedal curve of the parabola $t \mapsto (2at,\, t^2)$ with respect to the point $(0, b)$, where $a^2 \neq b$. Show that the nonparametric form of the pedal curve is

$$(x^2 + y^2)y + (a^2 - b)x^2 - 2by^2 + b^2 y = 0.$$

13. Prove Lemma 4.17.

14. An ordinary pendulum swings back and forth in a circular arc. The oscillations are not isochronous. To phrase it differently, the time it takes for the (circular) pendulum to go from its staring point to its lowest point will be almost, but not quite independent, of the height from which the pendulum is released. In 1680 Huygens[4] observed two important facts:

4  Christiaan Huygens (1629–1695). A leading Dutch scientist of the 17th century. As a mathematician he was a major precursor of Leibniz and Newton. His astronomical contributions include the discovery of the rings of Saturn. In 1656, Huygens patented the first pendulum clock.

(a) a pendulum that swings back and forth in an inverted cycloidial arc is isochronous, and (b) the involute of a cycloid is a cycloid. Combined, these facts show that the involute of an inverted cycloidial arc can be used to constrain a pendulum so that it moves in a cycloidial arc. Prove (b). For (a), see the end of Notebook 4.

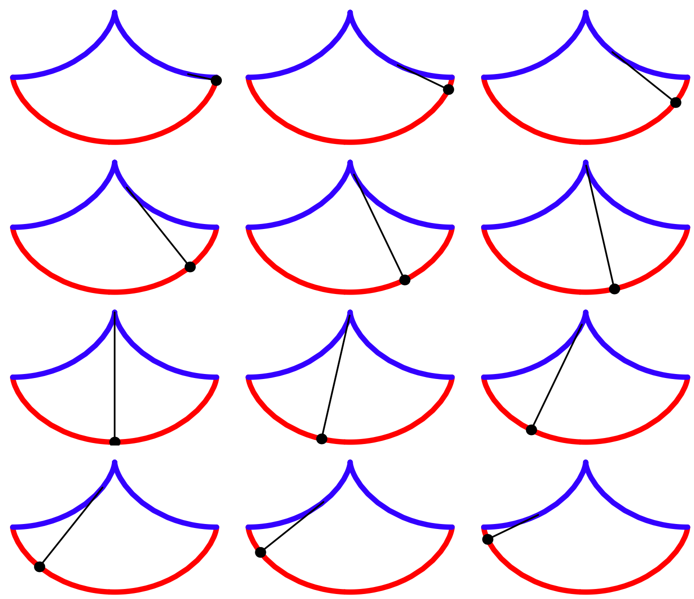

Figure 4.17: A cycloidal pendulum in action

15. Show that the formula for the pedal curve of $\boldsymbol{\alpha}$ with respect to the origin can be written as

$$\mathsf{pedal}[\boldsymbol{\alpha}, \mathbf{0}] = \frac{\boldsymbol{\alpha}\overline{\boldsymbol{\alpha}'} - \overline{\boldsymbol{\alpha}}\boldsymbol{\alpha}'}{2\overline{\boldsymbol{\alpha}'}}.$$

See [Zwi, page 150].

16. The ***contrapedal*** of a plane curve $\boldsymbol{\alpha}$ is defined analogously to the pedal of $\boldsymbol{\alpha}$. It is the locus of bases of perpendicular lines let down from a point $\mathbf{p}$ to a variable normal line to $\boldsymbol{\alpha}$. Prove that the exact formula is

$$\mathsf{contrapedal}[\boldsymbol{\alpha}, \mathbf{p}](t) = \mathbf{p} + \frac{\big(\boldsymbol{\alpha}(t) - \mathbf{p}\big) \cdot \boldsymbol{\alpha}'(t)}{\big\|\boldsymbol{\alpha}'(t)\big\|^2}\boldsymbol{\alpha}'(t).$$

M **17.** Show that the pedal curve of a circle with respect to a point on the circle is a cardioid. Plot pedal curves of a circle with respect to points inside the circle, and then with respect to points outside the circle.