Chapter 6

Global Properties of Plane Curves

The geometry of plane curves that we have been studying in the previous chapters has been local in nature. For example, the curvature of a plane curve describes the bending of that curve, point by point. In this chapter, we consider *global* properties that are concerned with the curve as a whole.

In Section 6.1, we define the *total signed curvature* of a plane curve $\boldsymbol{\alpha}$, by integrating its signed curvature $\boldsymbol{\kappa 2}$. The total signed curvature is an overall measure of curvature, directly related to the turning angle that was defined in Chapter 1. For a regular closed curve, the total signed curvature gives rise to a *turning number*, which is an integer. These concepts are well illustrated with reference to epitrochoids and hypotrochoids, curves formed by rolling wheels, described in Section 6.2. An example is given of relevant curvature functions.

Section 6.3 is devoted to the *rotation index* of a closed curve, which is defined topologically as the degree of an associated continuous mapping. We then show that it coincides with the turning number. The concept of *homotopy* for maps from a circle is introduced, and used to show that the turning number of a *simple* closed curve has absolute value 1.

Convex plane curves are considered in Section 6.4, where it is shown that a closed plane curve is convex if and only if its signed curvature does not change sign. We prove the *four vertex theorem* for such curves in Section 6.5, and illustrate it with the sine oval, parametrized using an iterated sine function.

Section 6.6 is concerned with more general *ovals*, closed curves for which $\boldsymbol{\kappa 2}$ not only does not change sign, but is either strictly positive or negative. We also establish basic facts about curves of *constant width*, and then give two classes of examples in Section 6.7. The formula for an oval in terms of its *support function* is derived in Section 6.8, using the envelope of a family of straight lines. A number of examples are investigated in the text and subsequent exercises.

6.1 Total Signed Curvature

The signed curvature $\boldsymbol{\kappa 2}$ of a plane curve $\boldsymbol{\alpha}$ was defined on page 14, and measures the bending of the curve at each of its points. A measure of the total bending of $\boldsymbol{\alpha}$ is given by an integral involving $\boldsymbol{\kappa 2}$.

Definition 6.1. *The* ***total signed curvature*** *of a curve* $\boldsymbol{\alpha}\colon [a,b] \to \mathbb{R}^2$ *is*

$$\mathsf{TSC}[\boldsymbol{\alpha}] = \int_a^b \boldsymbol{\kappa 2}[\boldsymbol{\alpha}](t) \big\| \boldsymbol{\alpha}'(t) \big\|\, dt,$$

where $\boldsymbol{\kappa 2}[\boldsymbol{\alpha}]$ *denotes the signed curvature of* $\boldsymbol{\alpha}$.

Although the definition depends upon the endpoints a, b, we assume that the mapping $\boldsymbol{\alpha}$ is in fact defined on an open interval I containing $[a,b]$.

First, let us check that the total signed curvature is a geometric concept.

Lemma 6.2. *The total signed curvature of a plane curve remains unchanged under a positive reparametrization, but changes sign under a negative one.*

Proof. Let $\boldsymbol{\gamma} = \boldsymbol{\alpha} \circ h$ where $\boldsymbol{\gamma}\colon (c,d) \to \mathbb{R}^2$ and $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^2$ are curves. We do the case when $h'(u) > 0$ for all u. Using Theorem 1.20 on page 16, and the formula from calculus for the change of variables in an integral, we compute

$$\begin{aligned}
\mathsf{TSC}[\boldsymbol{\alpha}] \;=\; \int_a^b \boldsymbol{\kappa 2}[\boldsymbol{\alpha}](t)\big\|\boldsymbol{\alpha}'(t)\big\|dt \;&=\; \int_c^d \boldsymbol{\kappa 2}[\boldsymbol{\alpha}]\big(h(u)\big)\big\|\boldsymbol{\alpha}'\big(h(u)\big)\big\|h'(u)\,du\\
&=\; \int_c^d \boldsymbol{\kappa 2}[\boldsymbol{\gamma}](u)\big\|\boldsymbol{\gamma}'(u)\big\|du\\
&=\; \mathsf{TSC}[\boldsymbol{\gamma}].
\end{aligned}$$

The proof that $\mathsf{TSC}[\boldsymbol{\gamma}] = -\mathsf{TSC}[\boldsymbol{\alpha}]$ when $\boldsymbol{\gamma}$ is a negative reparametrization of $\boldsymbol{\alpha}$ is similar. ▮

There is a simple relation linking the total signed curvature and the turning angle of a curve defined in Section 1.5.

Lemma 6.3. *The total signed curvature can be expressed in terms of the turning angle* $\boldsymbol{\theta}[\boldsymbol{\alpha}]$ *of* $\boldsymbol{\alpha}$ *by*

$$\mathsf{TSC}[\boldsymbol{\alpha}] = \boldsymbol{\theta}[\boldsymbol{\alpha}](b) - \boldsymbol{\theta}[\boldsymbol{\alpha}](a). \tag{6.1}$$

Proof. Equation (6.1) results when (1.22), page 20, is integrated. ▮

Since $\boldsymbol{\theta}[\boldsymbol{\alpha}]$ represents the direction of a unit tangent vector to the curve, $\mathsf{TSC}[\boldsymbol{\alpha}]$ measures the rotation of this vector. This is illustrated by the curve in Figure 6.2 for which $\boldsymbol{\theta}[\boldsymbol{\alpha}]$ varies between $-\pi/4$ and $5\pi/4$.

The total signed curvature of a *closed* curve is especially important. First, we define carefully the notion of closed curve.

Definition 6.4. *A regular curve* $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^n$ *is* ***closed*** *provided there is a constant* $c > 0$ *such that*

$$\boldsymbol{\alpha}(t+c) = \boldsymbol{\alpha}(t) \tag{6.2}$$

for all t*. The least such number* c *is called the* ***period*** *of* $\boldsymbol{\alpha}$*.*

Equation 6.2 expresses closure in the topological sense, as it ensures that $\boldsymbol{\alpha}$ determines a continuous mapping from a circle into $\mathbb{R}^n$, though we shall generally only consider closed curves that are regular. Exercise 1 provides a particular example of a nonregular curve satisfying (6.2).

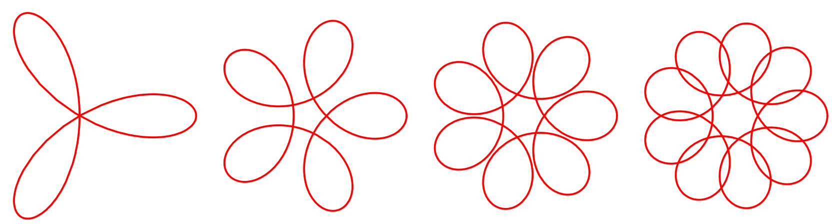

Figure 6.1: Hypotrochoids with turning numbers 2,4,6,8

Clearly, we can use (6.2) to define $\boldsymbol{\alpha}(t)$ for all t; that is, the domain of definition of a closed curve can be extended from (a,b) to $\mathbb{R}$. Just as in the case of a circle, the trace $\mathscr{C}$ of a regular closed curve $\boldsymbol{\alpha}$ is covered over and over again by $\boldsymbol{\alpha}$. Intuitively, when we speak of the length of a closed curve, we mean the length of the trace $\mathscr{C}$. Therefore, in the case of a closed curve we can use either $\mathbb{R}$, or a closed interval $[a,b]$, for the domain of definition of the curve, where $b-a$ is the period. Furthermore, we modify the definition of length given on page 9 as follows:

Definition 6.5. *Let* $\boldsymbol{\alpha}\colon \mathbb{R} \to \mathbb{R}^n$ *be a regular closed curve with period* c*. By the* ***length*** *of* $\boldsymbol{\alpha}$ *we mean the length of the restriction of* $\boldsymbol{\alpha}$ *to* $[0,c]$*, namely,*

$$\int_0^c \|\boldsymbol{\alpha}'(t)\|\, dt.$$

Next, we find the relation between the period and length of a closed curve.

Lemma 6.6. *Let* $\boldsymbol{\alpha}\colon\mathbb{R} \to \mathbb{R}^n$ *be a closed curve with period* c*, and let* $\boldsymbol{\beta}\colon\mathbb{R} \to \mathbb{R}^n$ *be a unit-speed reparametrization of* $\boldsymbol{\alpha}$*. Then* $\boldsymbol{\beta}$ *is also closed; the period of* $\boldsymbol{\beta}$ *is* L*, where* L *is the length of* $\boldsymbol{\alpha}$*.*

Proof. Let s denote the arc length function of $\boldsymbol{\alpha}$ starting at 0; we can assume (by Lemma 1.17, page 13) that $\boldsymbol{\alpha}(t) = \boldsymbol{\beta}\big(s(t)\big)$. We compute

$$\begin{aligned} s(t+c) \;=\; \int_0^{c+t} \big\|\boldsymbol{\alpha}'(u)\big\|\,du \;&=\; \int_0^{c} \big\|\boldsymbol{\alpha}'(u)\big\|\,du + \int_c^{c+t} \big\|\boldsymbol{\alpha}'(u)\big\|\,du \\ &=\; L + \int_c^{c+t} \big\|\boldsymbol{\alpha}'(u)\big\|\,du \\ &=\; L + \int_0^{t} \big\|\boldsymbol{\alpha}'(u)\big\|\,du \;=\; L + s(t). \end{aligned}$$

Thus

$$\boldsymbol{\beta}\big(s(t)+L\big) = \boldsymbol{\beta}\big(s(t+c)\big) = \boldsymbol{\alpha}(t+c) = \boldsymbol{\alpha}(t) = \boldsymbol{\beta}\big(s(t)\big),$$

for all $s(t)$; it follows that $\boldsymbol{\beta}$ is closed. Furthermore, since c is the least positive number such that $\boldsymbol{\alpha}(t+c) = \boldsymbol{\alpha}(c)$ for all t, it must be the case that L is the least positive number such that $\boldsymbol{\beta}(s+L) = \boldsymbol{\beta}(s)$ for all $\boldsymbol{\beta}$. ▮

We return to plane curves, that is, to the case $n = 2$.

Definition 6.7. *The* ***turning number*** *of a closed curve* $\boldsymbol{\alpha}\colon \mathbb{R} \to \mathbb{R}^2$ *is*

$$\mathsf{Turn}[\boldsymbol{\alpha}] = \frac{1}{2\pi}\int_0^c \boldsymbol{\kappa 2}[\boldsymbol{\alpha}](t)\big\|\boldsymbol{\alpha}'(t)\big\|\,dt,$$

where c denotes the period of $\boldsymbol{\alpha}$.

Thus the turning number of a closed curve $\boldsymbol{\alpha}$ is just the total signed curvature of $\boldsymbol{\alpha}$ divided by 2π, so that

$$\mathsf{TSC}[\boldsymbol{\alpha}] = 2\pi\mathsf{Turn}[\boldsymbol{\alpha}].$$

For example, let

$$\boldsymbol{\alpha}\colon [0, 2n\pi] \longrightarrow \mathbb{R}^2$$

be the function $t \mapsto a(\cos t, \sin t)$, so that $\boldsymbol{\alpha}$ covers a circle n times. It is easy to compute

$$\boldsymbol{\kappa 2}(\boldsymbol{\alpha}) = \frac{1}{a} \qquad \text{and} \qquad \big\|\boldsymbol{\alpha}'(t)\big\| = a.$$

Hence

$$\mathsf{TSC}[\boldsymbol{\alpha}] = 2n\pi \qquad \text{and} \qquad \mathsf{Turn}[\boldsymbol{\alpha}] = n$$

are independent of a.

More generally, if $\boldsymbol{\alpha}\colon \mathbb{R} \to \mathbb{R}^2$ is a regular closed curve with trace $\mathscr{C}$, then the mapping

$$t \mapsto \frac{\boldsymbol{\alpha}'(t)}{\big\|\boldsymbol{\alpha}'(t)\big\|} \tag{6.3}$$

gives rise to a mapping $\mathbf{\Phi}$ from $\mathscr{C}$ to the unit circle $S^1(1)$ of $\mathbb{R}^2$. Note that $\mathbf{\Phi}(\mathbf{p})$ is just the end point of the unit tangent vector to $\boldsymbol{\alpha}$ at $\mathbf{p}$. It is intuitively clear that when a point $\mathbf{p}$ goes around $\mathscr{C}$ once, its image $\mathbf{\Phi}(\mathbf{p})$ goes around $S^1(1)$ an integral number of times. The turning number of a *regular closed* curve should therefore be an integer; this will be proved rigorously in Section 6.3.

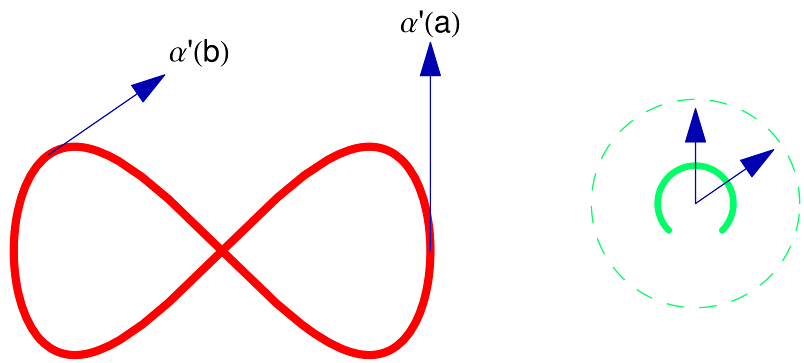

Figure 6.2: Turning of the tangent direction

If $\boldsymbol{\alpha}$ denotes the figure eight in Figure 6.2, then $\mathsf{Turn}[\boldsymbol{\alpha}] = 0$ because the 'clock hand' never reaches 6 pm. By contrast, each curve in Figure 6.1 has turning number equal to one less than the number of loops; a more explicit representation of a different example is displayed in Figure 6.4, after we have discussed the trochoids in more detail.

6.2 Trochoid Curves

The ***epitrochoid*** and ***hypotrochoid*** are good curves to illustrate turning number. They are defined by

$$\begin{aligned}\mathsf{epitrochoid}[a,b,h](t) &= \left((a+b)\cos t - h\cos\left(\frac{(a+b)t}{b}\right),\right.\\ &\qquad\left.(a+b)\sin t - h\sin\left(\frac{(a+b)t}{b}\right)\right)\end{aligned}$$

$$\begin{aligned}\mathsf{hypotrochoid}[a,b,h](t) &= \left((a-b)\cos t + h\cos\left(\frac{(a-b)t}{b}\right),\right.\\ &\qquad\left.(a-b)\sin t - h\sin\left(\frac{(a-b)t}{b}\right)\right).\end{aligned}$$

One can describe $\mathsf{epitrochoid}[a,b,h]$ as the parametrized curve that is traced out by a point $\mathbf{p}$ fixed relative to a circle of radius b rolling outside a fixed circle of radius a (see Figure 6.18 at the end of the chapter). Here h denotes the distance from $\mathbf{p}$ to the center of the rolling circle. In the case that $h = b$, the loops degenerate into points, and the resulting curve is a ***epicycloid*** (parametrized in Exercise 10 of Chapter 5).

Similarly, hypotrochoid$[a, b, h]$ is the curve traced out by a point $\mathbf{p}$ fixed relative to one circle (of radius b) rolling *inside* another circle (of radius a). The curves in Figure 6.1 are hypotrochoid$[2k-1, 1, k]$ for $k = 2, 3, 4, 5$, respectively. A ***hypocycloid*** is a hypotrochoid for which the loops degenerate to points, which again occurs for $h = b$. Abstractly speaking, the parametriztion of the hypotrochoid is obtained from that of the epitrochoid by simultaneously changing the signs of b and h (Exercise 6).

Both epitrochoid$[a, b, h]$ and hypotrochoid$[a, b, h]$ are precisely contained in a circle of radius $a + b + h$. Each has a/b loops as t ranges over the interval $[0, 2\pi]$ provided a/b is an integer. More generally, if a, b are integers, it can be verified experimentally that the number of loops is $a/\gcd(a, b)$, where $\gcd(a, b)$ denotes the greatest common denominator (or highest common factor) of a, b. Whilst the curve will eventually close up provided a/b is rational, it never closes if a/b is irrational; this can be exploited to draw pictures like Figure 6.3.

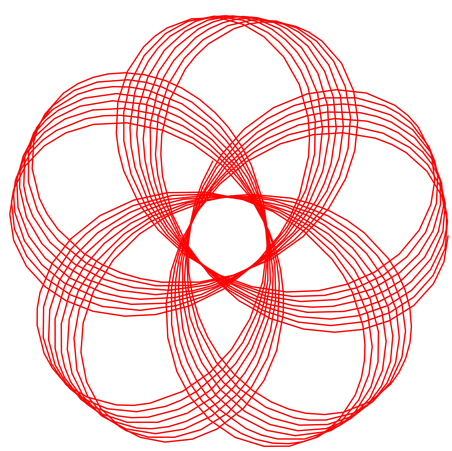

Figure 6.3: $t \mapsto$ epitrochoid$[7, \sqrt{2}, 8](t)$ with $0 \leqslant t \leqslant 16\pi$

As an example to introduce the next section, consider an epitrochoid with 5 inner loops, given by

$$\text{epitrochoid}[5, 1, 3](t) = \big(6\cos t - 3\cos 6t,\ 6\sin t - 3\sin 6t\big).$$

It is clear from Figure 6.4 that the curvature has constant sign. But it varies considerably, attaining its greatest (absolute) value during the inner loops. On the other hand, its total signed curvature (plotted rising steadily in Figure 6.5) is surprisingly linear, since the fluctuations of

$$t \mapsto \boldsymbol{\kappa 2}[\boldsymbol{\alpha}](t)\|\boldsymbol{\alpha}'(t)\|$$

(the base curve in Figure 6.5) are relatively small.

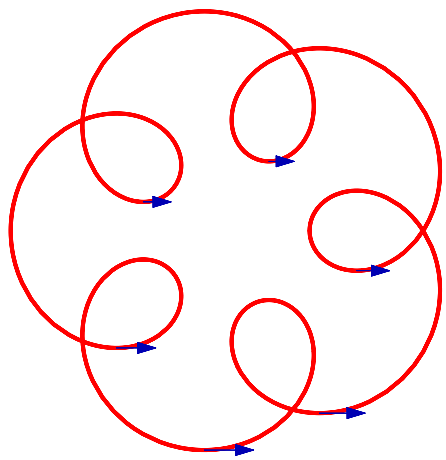

Figure 6.4: $\boldsymbol{\alpha} =$ epitrochoid$[5, 1, 3]$

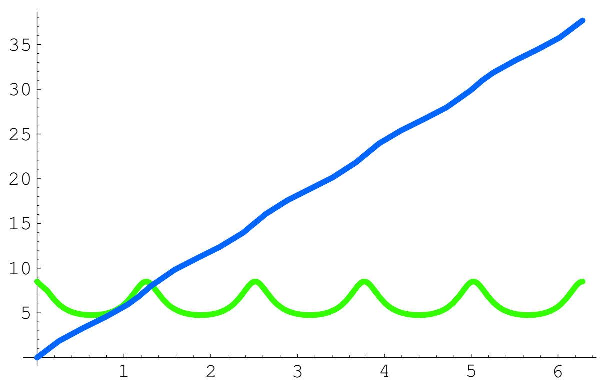

Figure 6.5: TSC$[\boldsymbol{\alpha}]$ and its integrand

Using the turning angle interpretation from Lemma 6.3, it is already clear from Figure 6.4 that $\boldsymbol{\alpha}$ has turning number 6. Indeed, the mini-arrows were positioned by solving an equation asserting that the tangent vector points in the direction of the positive x axis, and they show that the tangent vector undergoes 6 full turns in traversing the curve once. We shall now prove that the turning number of a regular closed curve is always an integer.

6.3 The Rotation Index of a Closed Curve

In Section 6.1, we gave the definition of turning number in terms of the total signed curvature. In this section, we determine the turning number more directly in terms of the mapping (6.3). First, an auxiliary definition.

Definition 6.8. *Let $\phi\colon S^1(1) \to S^1(1)$ be a continuous function, where $S^1(1)$ denotes a circle of radius 1 and center the origin in $\mathbb{R}^2$. Let $\widetilde{\phi}\colon \mathbb{R} \to \mathbb{R}$ be a continuous function such that*

$$\phi(\cos t,\, \sin t) = \big(\cos\widetilde{\phi}(t),\, \sin\widetilde{\phi}(t)\big) \tag{6.4}$$

*for all t. The **degree** of ϕ is the integer n such that $\widetilde{\phi}(2\pi) - \widetilde{\phi}(0) = 2\pi n$.*

The choice of $\widetilde{\phi}$ is not unique, since we can certainly add integer multiples of 2π to its value without affecting (6.4). But we have

Lemma 6.9. *The definition of degree is independent of the choice of $\widetilde{\phi}$.*

Proof. Let $\widehat{\phi}\colon \mathbb{R} \to \mathbb{R}$ be another continuous function satisfying the same conditions as $\widetilde{\phi}$. Then we have

$$\widehat{\phi}(t) - \widetilde{\phi}(t) = 2\pi n(t)$$

where $n(t)$ is an integer. Since $n(t)$ is continuous, it must be constant. Thus $\widetilde{\phi}(2\pi) - \widetilde{\phi}(0) = \widehat{\phi}(2\pi) - \widehat{\phi}(0)$. ▮

We next use the notion of degree of a map to define the rotation index of a curve. Let $\boldsymbol{\alpha}\colon \mathbb{R} \to \mathbb{R}^2$ be a regular closed curve. Rescale the parameter so that its period is 2π and $\boldsymbol{\alpha}$ determines a mapping with domain $S^1(1)$.

Definition 6.10. *The **rotation index** of $\boldsymbol{\alpha}$ is the degree of the corresponding mapping $\boldsymbol{\Phi}[\boldsymbol{\alpha}]\colon S^1(1) \to S^1(1)$ defined by*

$$\boldsymbol{\Phi}[\boldsymbol{\alpha}](t) = \frac{\boldsymbol{\alpha}'(t)}{\|\boldsymbol{\alpha}'(t)\|}. \tag{6.5}$$

Notice that the rotation index of a curve is defined *topologically*. By contrast, the turning number is defined *analytically*, as an integral. It is now easy for us to prove that these integers are the same.

Theorem 6.11. *The rotation index of a regular closed curve $\boldsymbol{\alpha}$ coincides with the turning number of $\boldsymbol{\alpha}$.*

Proof. Without loss of generality, we can assume that $\boldsymbol{\alpha}$ has unit speed. Then by (5.9) on page 137,

$$\boldsymbol{\Phi}[\boldsymbol{\alpha}](s) = \boldsymbol{\alpha}'(s) = \big(\cos\boldsymbol{\theta}(s),\, \sin\boldsymbol{\theta}(s)\big),$$

where $\boldsymbol{\theta} = \boldsymbol{\theta}[\boldsymbol{\alpha}]$ denotes the turning angle of $\boldsymbol{\alpha}$. Therefore, we can choose $\widetilde{\boldsymbol{\Phi}[\boldsymbol{\alpha}]} = \boldsymbol{\theta}$, and appeal to Lemma 6.3 and Corollary 1.27:

$$\begin{aligned}\mathsf{degree}\,\boldsymbol{\Phi}[\boldsymbol{\alpha}] &= \frac{1}{2\pi}\big(\boldsymbol{\theta}(2\pi) - \boldsymbol{\theta}(0)\big) = \frac{1}{2\pi}\int_0^{2\pi} \frac{d\boldsymbol{\theta}(s)}{ds}\,ds \\ &= \frac{1}{2\pi}\int_0^{2\pi} \boldsymbol{\kappa2}[\boldsymbol{\alpha}](s)\,ds \\ &= \mathsf{Turn}[\boldsymbol{\alpha}]. \quad \blacksquare\end{aligned}$$

Those closed curves which do not cross themselves form an important subclass.

Definition 6.12. *A regular closed curve $\boldsymbol{\alpha}\colon (a,b) \to \mathbb{R}^n$ with period c is* ***simple*** *provided $\boldsymbol{\alpha}(t_1) = \boldsymbol{\alpha}(t_2)$ if and only if $t_1 - t_2 = c$.*

For example, an ellipse is a simple closed curve, but the curves in Figure 6.1 are not.

It is clear intuitively that the rotation index of a simple closed curve is ± 1. To prove this rigorously, we need the important topological notion of homotopy. Let

$$\boldsymbol{\alpha}, \boldsymbol{\gamma}\colon [0, L] \longrightarrow X$$

be two continuous mappings such that

$$\boldsymbol{\alpha}(0) = \boldsymbol{\alpha}(L) \quad \text{and} \quad \boldsymbol{\gamma}(0) = \boldsymbol{\gamma}(L) \tag{6.6}$$

Here, X can be any topological space, although in our setting it suffices to take X to equal $\mathbb{R}^2$, so that $\boldsymbol{\alpha}, \boldsymbol{\gamma}$ may be thought of as curves. (Actually, in the proof below, the traces of the two curves will both be unit circles.) What is more, the condition (6.6) ensures that the domain of these curves can also be regarded as a circle (recall (6.2)).

Definition 6.13. *The mappings $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ are* ***homotopic*** *(as maps from a circle) if there exists a continuous mapping*

$$F\colon [0,1] \times [0, L] \longrightarrow X,$$

such that

(i) *$F(0,t) = \boldsymbol{\alpha}(t)$ for each t,*

(ii) *$F(1,t) = \boldsymbol{\gamma}(t)$ for each t,*

(iii) *$F(u,0) = F(u,L)$ for each u.*

The map F is called a ***homotopy*** *between $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$.*

Applying Lemma 6.8, we may define the degree of the mapping $t \to F(u,t)$ for each fixed u. But this map varies continuously with u, so its degree must in fact be constant, as in Lemma 6.9. In conclusion, *homotopic curves have the same degree*. This is used to prove the following theorem, due to H. Hopf[1] [Hopf1].

Theorem 6.14. *The turning number of a simple closed plane curve $\boldsymbol{\alpha}$ is ± 1.*

Proof. Fix a point $\mathbf{p}$ on the trace $\mathscr{C}$ of $\boldsymbol{\alpha}$ with the property that $\mathscr{C}$ lies entirely to one side of the tangent line at $\mathbf{p}$. This is always possible: choose a line that does not meet $\mathscr{C}$ and then translate it until it becomes tangent to $\mathscr{C}$. The idea of the proof is then to construct a homotopy between the unit tangent map $\boldsymbol{\Phi}[\boldsymbol{\alpha}]$ of (6.5), whose degree we know to be $\mathsf{Turn}[\boldsymbol{\alpha}]$, and a mapping of degree ± 1 defined by secants passing through $\mathbf{p}$.

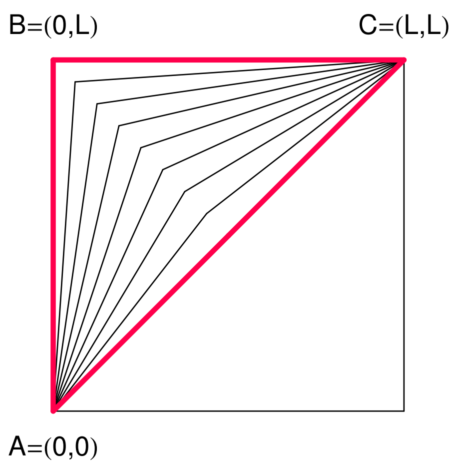

Figure 6.6: A homotopy square

Denote by L the length of $\boldsymbol{\alpha}$ and consider the triangular region

$$\mathcal{T} = \{\, (t_1, t_2) \;\mid\; 0 \leqslant t_1 \leqslant t_2 \leqslant L \,\}$$

shown in Figure 6.6. Let $\boldsymbol{\beta}$ be a reparametrization of $\boldsymbol{\alpha}$ with $\boldsymbol{\beta}(0) = \mathbf{p}$. The ***secant map*** $\Sigma\colon \mathcal{T} \to S^1(1)$ is defined by

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1  Heinz Hopf (1894–1971). Professor at the Eidgenössische Technische Hochschule in Zürich. The greater part of his work was in algebraic topology, motivated by an exceptional geometric intuition. In 1931, Hopf studied homotopy classes of maps from the sphere S^3 to the sphere S^2 and defined what is now known as the Hopf invariant.

$$\Sigma(t_1,t_2) = \begin{cases} \dfrac{\boldsymbol{\beta}'(t)}{\|\boldsymbol{\beta}'(t)\|} & \text{for } t_1 = t_2 = t, \\ -\dfrac{\boldsymbol{\beta}'(0)}{\|\boldsymbol{\beta}'(0)\|} & \text{for } t_1 = 0 \text{ and } t_2 = L \\ \dfrac{\boldsymbol{\beta}(t_2)-\boldsymbol{\beta}(t_1)}{\|\boldsymbol{\beta}(t_2)-\boldsymbol{\beta}(t_1)\|} & \text{otherwise.} \end{cases}$$

Since $\boldsymbol{\beta}$ is regular and simple, Σ is continuous. Let $A=(0,0)$, $B=(0,L)$ and $C=(L,L)$ be the vertices of $\mathcal{T}$, as in Figure 6.6. Because the restriction of Σ to the side AC is $\boldsymbol{\beta}'/\|\boldsymbol{\beta}'\|$, the degree of this restriction is the turning number of $\boldsymbol{\beta}$. Thus, by construction, $\boldsymbol{\beta}'/\|\boldsymbol{\beta}'\|$ is homotopic to the restriction of Σ to the path consisting of the sides AB and BC joined together. We must show that the degree of the latter map is ± 1.

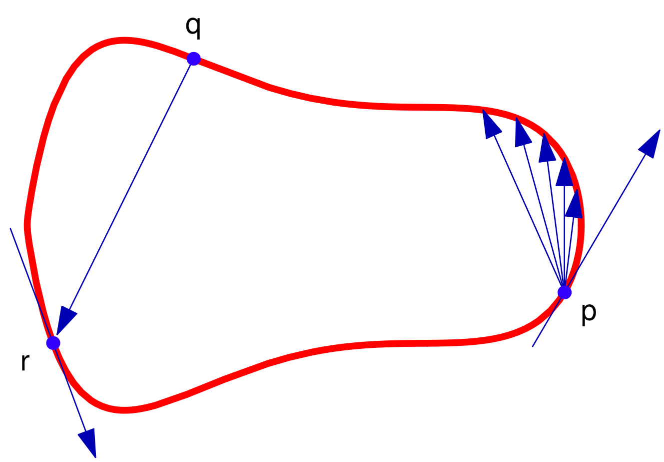

Figure 6.7: Secants and tangents

Assume that $\boldsymbol{\beta}$ is oriented with respect to $\mathbb{R}^2$, so that the angle from $\boldsymbol{\beta}'(0)$ to $-\boldsymbol{\beta}'(0)$ is π. The restriction of Σ to AB is represented by the family of unit vectors parallel to those (partially) shown emanating from $\mathbf{p}$ in Figure 6.7, and covers one half of $S^1(1)$, by the judicious choice of $\mathbf{p}$. Similarly, the restriction of Σ to BC covers the other half of $S^1(1)$. Hence the degree of Σ restricted to AB and BC is $+1$. Reversing the orientation, we obtain -1 for the degree. This completes the proof. ▮

Corollary 6.15. *If $\boldsymbol{\beta}$ is a simple closed unit-speed curve of period L, then the map $s \mapsto \boldsymbol{\beta}'(s)$ maps the interval $[0,L]$ onto all of the unit circle $S^1(1)$.*

There is actually a far-reaching generalization of Theorem 6.14:

Theorem 6.16. (Whitney-Graustein) *Two curves that have the same turning number are homotopic.*

For a proof of this theorem see [BeGo, page 325].

6.4 Convex Plane Curves

Any straight line ℓ divides $\mathbb{R}^2$ into two half-planes H_1 and H_2 such that

$$H_1 \cup H_2 = \mathbb{R}^2 \qquad \text{and} \qquad H_1 \cap H_2 = \ell.$$

We say that a curve $\mathscr{C}$ *lies on one side of* ℓ provided either $\mathscr{C}$ is completely contained in H_1 or $\mathscr{C}$ is completely contained in H_2.

Definition 6.17. A plane curve is ***convex*** *if it lies on one side of each of its tangent lines.*

Since the half-planes are closed, a straight line is certainly convex. We shall however be more concerned with closed curves in this section. Obviously, any ellipse is a convex curve, though we shall encounter many other examples. For a characterization of convex curves in terms of curvature, one needs the notion of a monotone function.

Definition 6.18. Let $f\colon (a, b) \to \mathbb{R}$ *be a function, not necessarily continuous. We say that* f *is* ***monotone increasing*** *provided that* $s \leqslant t$ *implies* $f(s) \leqslant f(t)$*, and* ***monotone decreasing*** *provided that* $s \leqslant t$ *implies* $f(s) \geqslant f(t)$*. If* f *is either monotone decreasing or monotone increasing, we say that* f *is* ***monotone****.*

It is easy to find examples of noncontinuous monotone functions, and also of continuous monotone functions that are not differentiable. In the differentiable case we have the following well-known result:

Lemma 6.19. A function $f\colon (a, b) \to \mathbb{R}$ *is monotone if and only if the derivative* f' *does not change sign on* (a, b)*. More precisely,* $f' \geqslant 0$ *implies monotone increasing and* $f' \leqslant 0$ *implies monotone decreasing.*

A glance at any simple closed convex curve $\mathscr{C}$ convinces us that the signed curvature of $\mathscr{C}$ does not change sign. We now prove this rigorously.

Theorem 6.20. A simple closed regular plane curve $\mathscr{C}$ *is convex if and only if its curvature* $\boldsymbol{\kappa 2}$ *has constant sign; that is,* $\boldsymbol{\kappa 2}$ *is either always nonpositive or always nonnegative.*

Proof. Parametrize $\mathscr{C}$ by a unit-speed curve $\boldsymbol{\beta}$ whose turning angle is $\boldsymbol{\theta} = \boldsymbol{\theta}[\boldsymbol{\beta}]$ (see Section 1.5). Since $\boldsymbol{\theta}' = \boldsymbol{\kappa 2}[\boldsymbol{\beta}]$, we must show that $\boldsymbol{\theta}$ is monotone if and only if $\boldsymbol{\beta}$ is convex, and then use Lemma 6.19.

Suppose that $\boldsymbol{\theta}$ is monotone, but that $\boldsymbol{\beta}$ is not convex. Then there exists a point $\mathbf{p}$ on $\boldsymbol{\beta}$ for which $\boldsymbol{\beta}$ lies on both sides of the tangent line ℓ to $\boldsymbol{\beta}$ at $\mathbf{p}$. Since $\boldsymbol{\beta}$ is closed, there are points $\mathbf{q}_1$ and $\mathbf{q}_2$ on opposite sides of ℓ that are farthest from ℓ.

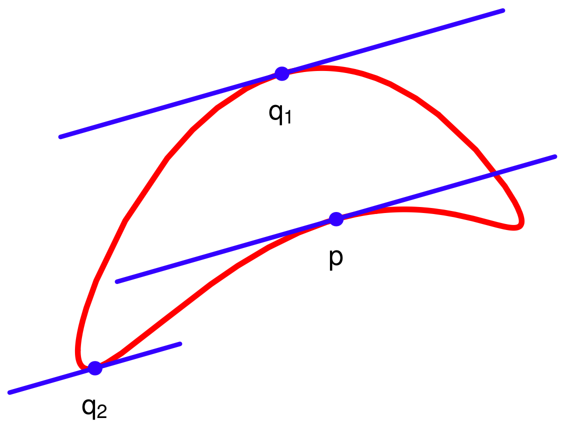

Figure 6.8: Parallel tangent lines on a nonconvex curve

The tangent lines ℓ_1 at $\mathbf{q}_1$ and ℓ_2 at $\mathbf{q}_2$ must be parallel to ℓ; see Figure 6.8. If this were not the case, we could construct a line $\tilde{\ell}$ through $\mathbf{q}_1$ (or $\mathbf{q}_2$) parallel to ℓ. Since $\tilde{\ell}$ would pass through $\mathbf{q}_1$ (or $\mathbf{q}_2$) but would not be tangent to $\boldsymbol{\beta}$, there would be points on $\boldsymbol{\beta}$ on both sides of $\tilde{\ell}$. There would then be points on $\boldsymbol{\beta}$ more distant from ℓ than $\mathbf{q}_1$ (or $\mathbf{q}_2$).

Two of the three points $\mathbf{p}, \mathbf{q}_1, \mathbf{q}_2$ must have tangents pointing in the same direction. In other words, if $\mathbf{p} = \boldsymbol{\beta}(s_0)$, $\mathbf{q}_1 = \boldsymbol{\beta}(s_1)$ and $\mathbf{q}_2 = \boldsymbol{\beta}(s_2)$, then there exist s_i and s_j with $s_i < s_j$ such that

$$\boldsymbol{\beta}'(s_i) = \boldsymbol{\beta}'(s_j) \qquad \text{and} \qquad \boldsymbol{\theta}(s_j) = \boldsymbol{\theta}(s_i) + 2n\pi,$$

for some integer n. Since $\boldsymbol{\theta}$ is monotone, Theorem 6.14 implies that $n = 0$, 1 or -1. If $n = 0$, then $\boldsymbol{\theta}(s_i) = \boldsymbol{\theta}(s_j)$, and the monotonicity of $\boldsymbol{\theta}$ implies that $\boldsymbol{\theta}$ is constant on the interval $[s_i, s_j]$. If $n = \pm 1$, then $\boldsymbol{\theta}$ is constant on the intervals $[0, s_i]$ and $[s_j, L]$. In either case, one of the segments of $\boldsymbol{\beta}$ between $\boldsymbol{\beta}(s_i)$ and $\boldsymbol{\beta}(s_j)$ is a straight line. Hence the tangent lines at $\boldsymbol{\beta}(s_i)$ and $\boldsymbol{\beta}(s_j)$ coincide. But ℓ, ℓ_1 and ℓ_2 are distinct. Thus we reach a contradiction, and so $\boldsymbol{\beta}$ must be convex.

To prove the converse, assume that $\boldsymbol{\beta}$ is convex, but that the turning angle $\boldsymbol{\theta}$ is not monotone. Then we can find s_0, s_1 and s_2 such that $s_1 < s_0 < s_2$ with

$\boldsymbol{\theta}(s_1) = \boldsymbol{\theta}(s_2) \neq \boldsymbol{\theta}(s_0)$. Corollary 6.15 says that $s \mapsto \boldsymbol{\beta}'(s)$ maps the interval $[0, L]$ onto all of the unit circle $S^1(1)$; hence there is s_3 such that $\boldsymbol{\beta}'(s_3) = -\boldsymbol{\beta}'(s_1)$. If the tangent lines at $\boldsymbol{\beta}(s_1)$, $\boldsymbol{\beta}(s_2)$ and $\boldsymbol{\beta}(s_3)$ are distinct, they are parallel, and one lies between the other two. This cannot be the case, since $\boldsymbol{\beta}$ is convex. Thus two of the tangent lines coincide, and there are points $\mathbf{p}$ and $\mathbf{q}$ of $\boldsymbol{\beta}$ lying on the same tangent line.

We show that the curve $\boldsymbol{\beta}$ is a straight line connecting $\mathbf{p}$ and $\mathbf{q}$. Let $\ell(\mathbf{p}, \mathbf{q})$ denote the straight line segment from $\mathbf{p}$ to $\mathbf{q}$. Suppose that some point $\mathbf{r}$ of $\ell(\mathbf{p}, \mathbf{q})$ is not on $\boldsymbol{\beta}$. Let $\widehat{\ell}$ be the straight line perpendicular to $\ell(\mathbf{p}, \mathbf{q})$ at $\mathbf{r}$. Since $\boldsymbol{\beta}$ is convex, $\widehat{\ell}$ is nowhere tangent to $\boldsymbol{\beta}$. Thus $\widehat{\ell}$ intersects $\boldsymbol{\beta}$ in at least two points $\mathbf{r}_1$ and $\mathbf{r}_2$. If $\mathbf{r}_1$ denotes the point closer to $\mathbf{r}$, then the tangent line to $\boldsymbol{\beta}$ at $\mathbf{r}_1$ has $\mathbf{r}_2$ on one side and one of $\mathbf{p}, \mathbf{q}$ on the other, contradicting the assumption that $\boldsymbol{\beta}$ is convex.

Hence $\mathbf{r}$ cannot exist, and so the straight line segment $\ell(\mathbf{p}, \mathbf{q})$ is contained in the trace of $\boldsymbol{\beta}$. Thus $\mathbf{p}$ and $\mathbf{q}$ are $\boldsymbol{\beta}(s_1)$ and $\boldsymbol{\beta}(s_2)$, so that the restriction of $\boldsymbol{\beta}$ to the interval $[s_1, s_2]$ is a straight line. Therefore, $\boldsymbol{\theta}$ is constant on $[s_1, s_2]$, and the assumption that $\boldsymbol{\theta}$ is not monotone leads to a contradiction. It follows that $\boldsymbol{\kappa 2}[\boldsymbol{\beta}]$ has constant sign. ▮

The proof of Theorem 6.20 contains that of the following result.

Corollary 6.21. *Let $\boldsymbol{\alpha}$ be a regular simple closed curve with turning angle $\boldsymbol{\theta}[\boldsymbol{\alpha}]$. If $\boldsymbol{\theta}[\boldsymbol{\alpha}](t_1) = \boldsymbol{\theta}[\boldsymbol{\alpha}](t_2)$ with $t_1 < t_2$, then the restriction of $\boldsymbol{\alpha}$ to the interval $[t_1, t_2]$ is a straight line.*

6.5 The Four Vertex Theorem

In this section, we prove a celebrated global theorem about plane curves. To understand the result, let us first consider an example.

The ***sine oval curve*** is defined by

$$\mathsf{sinoval}[n, a](t) = \big(a\cos t,\ a\sin^{(n)}(t)\big),$$

where $\sin^{(n)}(t)$ denotes the application

$$\underbrace{\sin(\sin\cdots(\sin\, t))}_{n}$$

of the sine function n times. (This iterated function is easily computable, as we shall see in Notebook 6.) Clearly, $\mathsf{sinoval}[1, a]$ is a circle of radius a. Next, $\mathsf{sinoval}[2, a]$ is the curve

$$t \mapsto \big(a\cos t,\ a\sin(\sin t)\big),$$

and so forth. As n increases, the top and bottom of $\mathsf{sinoval}[n, a]$ are pushed together more and more. A typical plot is shown in Figure 6.9.

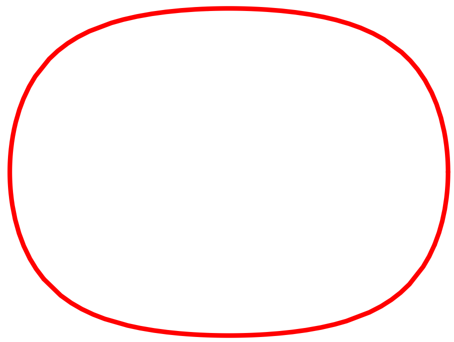

Figure 6.9: sinoval$[3, 1]$

The curvature of the sine oval has four maxima (all absolute) and four minima (two absolute and two local), as shown in the curvature graph in Figure 6.10.

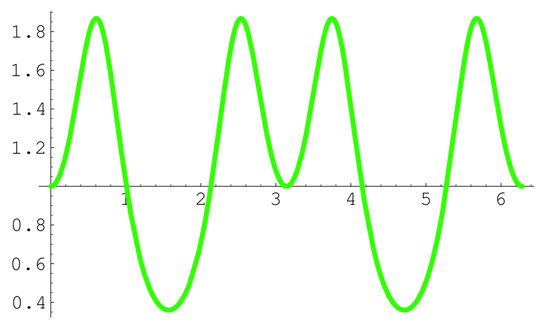

Figure 6.10: Curvature of sinoval$[3, 1]$

A simpler example is an ellipse. The curvature of an ellipse has exactly 2 maxima and 2 minima (see Exercise 9). Further examples lead naturally to the conjecture that the signed curvature of any simple closed convex curve has at least two maxima and two minima.

To prove this conjecture, we first make the following definition.

Definition 6.22. *A **vertex** of a regular plane curve is a point where the signed curvature has a relative maximum or minimum.*

On any closed curve, the continuous function $\boldsymbol{\kappa 2}$ must attain a maximum and a minimum, so there are at least two vertices, and they come in pairs. To find the vertices of a simple closed convex curve, we must determine those points where the derivative of the curvature vanishes. It follows from differentiating equation (1.15), page 16, that the definition of vertex is independent of the choice of regular parametrization.

We need an elementary lemma:

Lemma 6.23. *Let ℓ be a line in the plane. Then there exist constant vectors $\mathbf{a}, \mathbf{c} \in \mathbb{R}^2$ with $\mathbf{c} \neq 0$ such that $\mathbf{z} \in \ell$ if and only if $(\mathbf{z} - \mathbf{a}) \cdot \mathbf{c} = 0$.*

Proof. If we parametrize ℓ by $\boldsymbol{\alpha}(t) = \mathbf{p} + t\mathbf{q}$, we can take $\mathbf{a} = \mathbf{p}$ and $\mathbf{c} = J\mathbf{q}$. ∎

Finally, we are ready to prove the conjecture.

Theorem 6.24. (Four Vertex Theorem) *A simple closed convex curve $\boldsymbol{\alpha}$ has at least four vertices.*

Proof. The derivative $\boldsymbol{\kappa 2}'$ vanishes at each vertex of $\boldsymbol{\alpha}$. If $\boldsymbol{\kappa 2}$ is constant on any segment of $\boldsymbol{\alpha}$, then every point on the segment is a vertex, and we are done. We can therefore assume that $\boldsymbol{\alpha}$ contains neither circular arcs nor straight line segments, and that $\boldsymbol{\alpha}$ has at least two distinct vertices $\mathbf{p}, \mathbf{q} \in \mathbb{R}^2$; without loss of generality, $\boldsymbol{\alpha}(0) = \mathbf{p}$. We now show that the assumption that $\mathbf{p}$ and $\mathbf{q}$ are the only vertices leads to a contradiction. Because vertices come in pairs, this will complete the proof of the theorem.

Let ℓ be the straight line joining $\mathbf{p}$ and $\mathbf{q}$; then ℓ divides $\boldsymbol{\alpha}$ into two segments. Since we have assumed that there are exactly two vertices, it must be the case that $\boldsymbol{\kappa 2}'$ is positive on one segment of $\boldsymbol{\alpha}$ and negative on the other. Lemma 6.23 says that there are constant vectors $\mathbf{a}$ and $\mathbf{c} \neq 0$ such that $\mathbf{z} \in \ell$ if and only if $(\mathbf{z} - \mathbf{a}) \cdot \mathbf{c} = 0$. Because $\boldsymbol{\alpha}$ is convex, $(\mathbf{z} - \mathbf{a}) \cdot \mathbf{c}$ is positive on one segment of $\boldsymbol{\alpha}$ and negative on the other.

It can be checked case by case that $\boldsymbol{\kappa 2}'(s)\big(\boldsymbol{\alpha}(s) - \mathbf{a}\big) \cdot \mathbf{c}$ does not change sign on $\boldsymbol{\alpha}$. Hence there must be an s_0 for which $\boldsymbol{\kappa 2}'(s_0)\big(\boldsymbol{\alpha}(s_0) - \mathbf{a}\big) \cdot \mathbf{c} \neq 0$, and so the integral of $\boldsymbol{\kappa 2}'(s)\big(\boldsymbol{\alpha}(s_0) - \mathbf{a}\big) \cdot \mathbf{c}$ from 0 to L is nonzero, where L is the length of $\mathscr{C}$. Integrating by parts, we obtain

$$
\begin{aligned}
0 &\neq \int_0^L \boldsymbol{\kappa 2}'(s)\big(\boldsymbol{\alpha}(s) - \mathbf{a}\big) \cdot \mathbf{c}\, ds \\
&= \boldsymbol{\kappa 2}(s)\big(\boldsymbol{\alpha}(s) - \mathbf{a}\big) \cdot \mathbf{c}\,\Big|_0^L - \int_0^L \boldsymbol{\kappa 2}(s)\big(\boldsymbol{\alpha}'(s) \cdot \mathbf{c}\big) ds \\
&= \int_0^L \big(-\boldsymbol{\kappa 2}(s)\boldsymbol{\alpha}'(s) \cdot \mathbf{c}\big) ds.
\end{aligned}
\tag{6.7}
$$

Lemma 1.21, page 16, implies that $J\boldsymbol{\alpha}''(s) = -\boldsymbol{\kappa 2}(s)\boldsymbol{\alpha}'(s)$, and so the last integral of (6.7) can be written as

$$\int_0^L \big(\boldsymbol{\alpha}''(s) \cdot \mathbf{c}\big) ds = \boldsymbol{\alpha}'(s) \cdot \mathbf{c}\,\Big|_0^L.$$

But $\boldsymbol{\alpha}'(L) = \boldsymbol{\alpha}'(0)$, so we reach a contradiction. It follows that the assumption that $\mathscr{C}$ had only two vertices is false. ∎

It turns out that any simple closed curve, convex or not, has at least four vertices; this is the result of Mukhopadhyaya [Muk]. On the other hand, it is easy to find nonsimple closed curves with only two vertices; see Exercise 5.

6.6 Curves of Constant Width

Why is a manhole cover (at least in the United States) round? Probably a square manhole cover would be easier to manufacture. But a square manhole cover, when rotated, could slip through the manhole; however, a circular manhole cover can never slip underground. The reason is that a circular manhole cover has constant width, but the width of a square manhole cover varies between a and $a\sqrt{2}$, where a is the length of a side.

Are there other curves of constant width? This question was answered affirmatively by Euler [Euler4] over two hundred years ago. A city governed by a mathematician might want to use manhole covers in the shape of a Reuleaux triangle or the involute of a deltoid, both to be discussed in Sections 6.7.

In this section we concentrate on the basic theory of curves of constant width. The literature on this subject is large: see, for example, [Bar], [Bieb, pages 27-29], [Dark], [Euler4], [Fischer, chapter 4], [HC-V, page 216], [MiPa, pages 66-71], [RaTo1, pages 137-150], [Strub, volume 1, pages 120-124] and [Stru2, pages 47-51].

Definition 6.25. *An **oval** is a simple closed plane curve for which the signed curvature $\boldsymbol{\kappa}\mathbf{2}$ is always strictly positive or always strictly negative.*

By Theorem 6.20, an oval is convex. The converse is false; for example, the curve $x^4 + y^4 = 1$ has points with vanishing curvature (see Notebook 6). If $\boldsymbol{\alpha}$ is an oval whose signed curvature is always negative, consider instead $t \mapsto \boldsymbol{\alpha}(-t)$; thus the signed curvature of an oval can be assumed to be positive.

Let $\boldsymbol{\beta}\colon \mathbb{R} \to \mathbb{R}^2$ be a unit-speed oval and let $\boldsymbol{\beta}(s)$ be a point on $\boldsymbol{\beta}$. Corollary 6.15 implies that there is a point $\boldsymbol{\beta}(s^{\mathrm{o}})$ on $\boldsymbol{\beta}$ for which $\boldsymbol{\beta}'(s^{\mathrm{o}}) = -\boldsymbol{\beta}'(s)$, and the reasoning in the proof of Theorem 6.20 implies that $\boldsymbol{\beta}(s^{\mathrm{o}})$ is unique. We call $\boldsymbol{\beta}(s^{\mathrm{o}})$ the ***point opposite to*** $\boldsymbol{\beta}(s)$. Let $\widehat{\boldsymbol{\beta}}\colon \mathbb{R} \to \mathbb{R}^2$ be the curve such that $\widehat{\boldsymbol{\beta}}(s) = \boldsymbol{\beta}(s^{\mathrm{o}})$ for all s. Let us write $\mathbf{T}(s) = \boldsymbol{\beta}'(s)$. Since $\mathbf{T}(s)$ and $J\mathbf{T}(s)$ are linearly independent, we can write

$$\widehat{\boldsymbol{\beta}}(s) - \boldsymbol{\beta}(s) = \lambda(s)\mathbf{T}(s) + \mu(s)J\mathbf{T}(s), \tag{6.8}$$

where $\lambda, \mu\colon \mathbb{R} \to \mathbb{R}$ are piecewise-differentiable functions. Note that even though $\boldsymbol{\beta}$ has unit-speed, $\widehat{\boldsymbol{\beta}}$ may not have unit-speed. We put $\widehat{\mathbf{T}}(s) = \widehat{\boldsymbol{\beta}}'(s)/\|\widehat{\boldsymbol{\beta}}'(s)\|$.

Definition 6.26. *The **spread** and **width** of a unit-speed oval $\boldsymbol{\beta}\colon \mathbb{R} \to \mathbb{R}^2$ are the functions λ and μ. We say that an oval has **constant width** if μ is constant.*

Figure 6.11 shows that $|\lambda(s)|$ (respectively $|\mu(s)|$) is the distance between the normal (respectively, tangent) lines at $\boldsymbol{\beta}(s)$ and $\widehat{\boldsymbol{\beta}}(s)$. Moreover, $\sqrt{\lambda(s)^2 + \mu(s)^2}$ is the distance between the two opposite points.

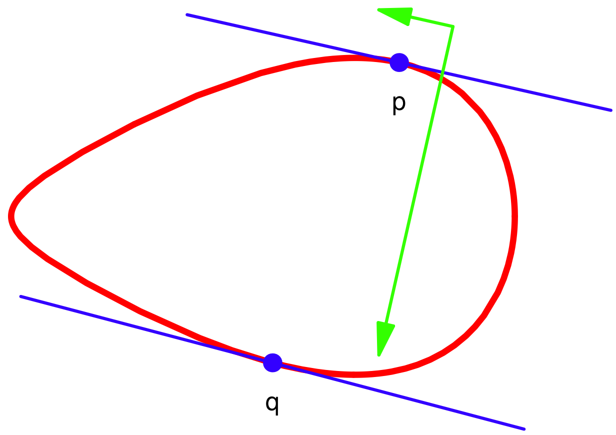

Figure 6.11: Spread and width

Next, we prove a fundamental theorem about constant width ovals.

Theorem 6.27. (Barbier[2]) *An oval of constant width w has length πw.*

Proof. We parametrize the oval by a unit-speed curve $\boldsymbol{\beta}\colon \mathbb{R} \to \mathbb{R}^2$. From (6.8) and Lemma 1.21, it follows that

$$\widehat{\boldsymbol{\beta}}' = \left(1 + \frac{d\lambda}{ds} - \mu\boldsymbol{\kappa}\mathbf{2}\right)\mathbf{T} + \left(\lambda\boldsymbol{\kappa}\mathbf{2} + \frac{d\mu}{ds}\right)J\mathbf{T}. \tag{6.9}$$

On the other hand, if $\widehat{s}$ denotes the arc length function of $\widehat{\boldsymbol{\beta}}$ we have

$$\frac{\widehat{\boldsymbol{\beta}}'}{\|\widehat{\boldsymbol{\beta}}'\|} = -\mathbf{T} \qquad \text{and} \qquad \frac{d\widehat{s}}{ds} = \|\widehat{\boldsymbol{\beta}}'\|,$$

so that

$$\widehat{\boldsymbol{\beta}}' = -\frac{d\widehat{s}}{ds}\mathbf{T}. \tag{6.10}$$

From (6.9) and (6.10) we obtain

$$\left(1 + \frac{d\widehat{s}}{ds} + \frac{d\lambda}{ds} - \mu\boldsymbol{\kappa}\mathbf{2}\right)\mathbf{T} + \left(\lambda\boldsymbol{\kappa}\mathbf{2} + \frac{d\mu}{ds}\right)J\mathbf{T} = 0. \tag{6.11}$$

Let $\vartheta = \boldsymbol{\theta} = \boldsymbol{\theta}[\boldsymbol{\beta}]$ denote the turning angle, so that

$$\boldsymbol{\kappa}\mathbf{2} = \frac{d\vartheta}{ds}.$$

[2]Joseph Émile Barbier (1839–1889). French mathematician who wrote many excellent papers on differential geometry, number theory and probability.

We can use this to rewrite (6.11) as

$$\left(\frac{d(s+\widehat{s})}{ds}+\frac{d\lambda}{ds}-\mu\frac{d\vartheta}{ds}\right)\mathbf{T}+\left(\lambda\frac{d\vartheta}{ds}+\frac{d\mu}{ds}\right)J\mathbf{T}=0,$$

from which we conclude that

$$\frac{d(s+\widehat{s})}{ds}+\frac{d\lambda}{ds}-\mu\frac{d\vartheta}{ds}=0 \qquad \text{and} \qquad \lambda\frac{d\vartheta}{ds}+\frac{d\mu}{ds}=0. \tag{6.12}$$

Now suppose that μ has the constant value w. Since the curvature of $\boldsymbol{\beta}$ is always positive, the second equation of (6.12) tells us that $\lambda = 0$, and so the first equation of (6.12) reduces to

$$\frac{d(s+\widehat{s})}{ds}-w\frac{d\vartheta}{ds}=0. \tag{6.13}$$

Fix a point $\mathbf{p}$ on the oval, and let s_0 and s_1 be such that $\boldsymbol{\beta}(s_0)=\mathbf{p}$ and $\boldsymbol{\beta}(s_1)=\widehat{\mathbf{p}}$, where $\widehat{\mathbf{p}}$ is the point on the oval opposite to $\mathbf{p}$. If L denotes the length of the oval, we have from (6.13) that

$$L=\int_{s_0}^{s_1}\frac{d(s+\widehat{s})}{ds}\,ds=\int_{s_0}^{s_1}w\frac{d\vartheta}{ds}\,ds=w\int_0^{\pi}d\vartheta=w\pi. \blacksquare$$

An elegant generalization of Barbier's theorem, giving formulas for the width and spread of a general oval, has been proved by Mellish[3].

Theorem 6.28. (Mellish) *The width μ of an oval parametrized by a unit-speed curve $\boldsymbol{\beta}$, as a function of ϑ, is a solution of the differential equation*

$$\frac{d^2\mu}{d\vartheta^2}+\mu=f(\vartheta), \tag{6.14}$$

where

$$f(\vartheta)=\frac{1}{\boldsymbol{\kappa 2}(\vartheta)}+\frac{1}{\boldsymbol{\kappa 2}(\vartheta+\pi)}.$$

Moreover, if we set

$$U(c)=\int_0^c f(t)\cos t\,dt, \qquad V(c)=\int_0^c f(t)\sin t\,dt,$$

then

$$\mu(\vartheta)=\big(U(\vartheta)-\tfrac12 U(\pi)\big)\sin\vartheta-\big(V(\vartheta)-\tfrac12 V(\pi)\big)\cos\vartheta.$$

Proof. We can rewrite (6.12) in terms of differentials:

$$ds+d\widehat{s}+d\lambda-\mu\,d\vartheta=0 \qquad \text{and} \qquad \lambda\,d\vartheta+d\mu=0.$$

[3] Arthur Preston Mellish (1905–1930). Canadian mathematician.

Clearly, this implies that

$$\frac{ds}{d\vartheta}+\frac{d\widehat{s}}{d\vartheta}+\frac{d\lambda}{d\vartheta}-\mu=0 \qquad \text{and} \qquad \lambda+\frac{d\mu}{d\vartheta}=0. \tag{6.15}$$

But

$$\frac{ds}{d\vartheta}=\frac{1}{\boldsymbol{\kappa 2}(\vartheta)} \qquad \text{and} \qquad \frac{d\widehat{s}}{d\vartheta}=\frac{1}{\boldsymbol{\kappa 2}(\vartheta+\pi)},$$

so that (6.15) becomes

$$f(\vartheta)+\frac{d\lambda}{d\vartheta}-\mu=0 \qquad \text{and} \qquad \lambda+\frac{d\mu}{d\vartheta}=0. \tag{6.16}$$

Elimination of λ in (6.16) yields (6.14).

The general solution of (6.14) is

$$\mu(\vartheta)=\sin\vartheta\left(\int_0^{\vartheta} f(t)\cos t\,dt+C_1\right)-\cos\vartheta\left(\int_0^{\vartheta} f(t)\sin t\,dt+C_2\right),$$

and the arbitrary constants C_1 and C_2 can be determined by observing that μ,λ,f are all periodic functions of ϑ with period π. ▮

There is a similar formula for the spread $\lambda(\vartheta)$ of the oval (see Exercise 13).

6.7 Reuleaux Polygons and Involutes

If we relax the condition that our closed curves be regular, simple examples with constant width can be constructed from regular polygons with odd numbers of sides. Let $\mathsf{P}[n,a]$ be a regular polygon with $2n+1$ sides, where a denotes the length of any side. Corresponding to each vertex $\mathbf{p}$, there is a side of $\mathsf{P}[n,a]$ that is most distant from $\mathbf{p}$. Let $\mathbf{p}_1$ and $\mathbf{p}_2$ be its vertices, and let $\widehat{\mathbf{p}_1\mathbf{p}_2}$ be the arc of the circle with center $\mathbf{p}$ connecting $\mathbf{p}_1$ and $\mathbf{p}_2$.

Definition 6.29. *The* ***Reuleaux***[4] ***polygon*** *is the curve* $\mathsf{R}[n,a]$ *made up of the circular arcs* $\widehat{\mathbf{p}_1\mathbf{p}_2}$ *formed when* $\mathbf{p}$ *ranges over the vertices of* $\mathsf{P}[n,a]$.

The Reuleaux polygon $\mathsf{R}[n,a]$ consists of $2n+1$ arcs of a circle of radius a, each subtending an angle of $\pi/(2n+1)$; thus the length of $\mathsf{R}[n,a]$ equals πa. If we parametrize $\mathsf{R}[n,a]$ by arc length s, the associated mapping $\boldsymbol{\Phi}\colon S^1(1)\to S^1(1)$ (recall (6.3)) is undefined at the points $s=k\pi a/(2n+1)$ with $k=0,1,\dots,2n$. Ignoring this finite number of points, the images of $\boldsymbol{\Phi}$ and $-\boldsymbol{\Phi}$ exactly cover the circle, and *no* point has an opposite in the sense of the previous section! Despite this defect, we may still assert that the width of the convex curve $\mathsf{R}[n,a]$ is constant and equal to a. For however we orient the 'manhole', it will always fit snugly into a pipe of diameter a.

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[4] Franz Reuleaux, (1829–1905). German professor of machine design.

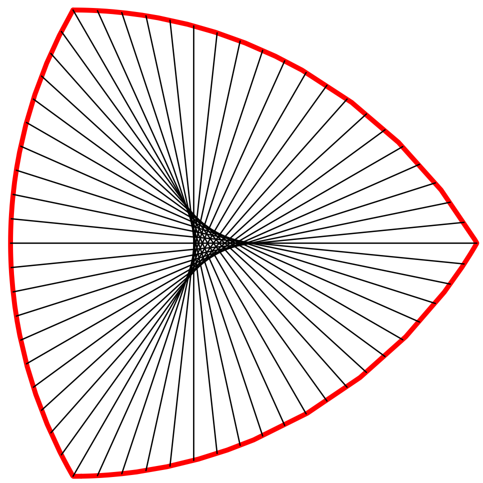

Figure 6.12: The Reuleaux triangle $\mathsf{R}[1,1]$ and a family of lines

The unit Reuleaux triangle is shown in Figure 6.12, together with straight lines joining points of the curve an arc length $\pi/2$ apart. The curve $\mathsf{R}[1,1]$ is the model for a cross section of the rotor in the Wankel[5] engine.

Curves with constant width can be effectively constructed as the involutes of a suitable curve. We shall illustrate this in terms of the deltoid, a special hypocycloid first defined in Exercise 2 of Chapter 2 on page 57.

Let $\mathscr{D}$ be a deltoid with vertices $\mathbf{a},\mathbf{b},\mathbf{c}$, together with a flexible cord attached to the curved side $\widehat{\mathbf{bc}}$. Keep the cord attached to a point $\mathbf{p}$ on $\widehat{\mathbf{bc}}$, and let both ends of the cord unwind. The end of the cord that was originally attached to $\mathbf{b}$ traces out a curve, part of $\mathscr{D}$'s involute, and similarly for the end of the cord attached to $\mathbf{c}$. Let $\mathbf{b}$ move to $\mathbf{q}$ and $\mathbf{c}$ to $\mathbf{r}$, and denote by $\overline{\mathbf{pq}}$ and $\overline{\mathbf{pr}}$ the line segments from $\mathbf{p}$ to $\mathbf{q}$ and from $\mathbf{p}$ to $\mathbf{r}$. By definition of the involute,

$$\mathsf{length}\,\overline{\mathbf{pq}} = \mathsf{length}\,\widehat{\mathbf{pb}} \quad \text{and} \quad \mathsf{length}\,\overline{\mathbf{pr}} = \mathsf{length}\,\widehat{\mathbf{pc}},$$

where $\widehat{\mathbf{pb}}$ and $\widehat{\mathbf{pc}}$ denote arcs of $\mathscr{D}$.

[5]Felix Heinrch Wankel (1902–1988). German engineer. The Wankel engine differs greatly from conventional engines. It retains the familiar intake, compression, power, and exhaust cycle but uses a rotor in the shape of a Reuleaux triangle, instead of a piston, cylinder, and mechanical valves. The Wankel engine has 40 percent fewer parts and roughly one third the bulk and weight of a comparable reciprocating engine. Within the Wankel, three chambers are formed by the sides of the rotor and the wall of the housing. The shape, size, and position of these chambers are constantly altered by the rotor's clockwise rotation. The engine is unique in that the power impulse is spread over approximately 270° degrees of crank shaft rotation, as compared to 180° degrees for the conventional reciprocating two-stroke engine.

Since the line segments $\overline{\mathbf{pq}}$ and $\overline{\mathbf{pr}}$ are both tangent to the deltoid at $\mathbf{p}$, they are part of the straight line segment $\overline{\mathbf{qr}}$ connecting $\mathbf{q}$ to $\mathbf{r}$. Consequently,

$$\text{length } \overline{\mathbf{qr}} = \text{length along the deltoid of } \widehat{\mathbf{bc}}.$$

Therefore, $\overline{\mathbf{qr}}$ has the same length, no matter the position of $\mathbf{p}$. It follows that the involute of $\mathscr{D}$ has constant width. This construction was originally carried out by Euler [Euler4] in more generality.

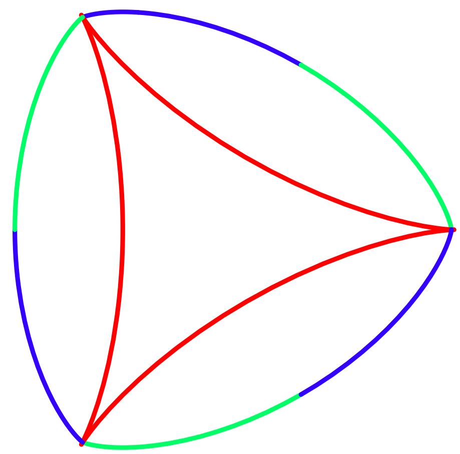

Figure 6.13: A deltoid and its involute

The involute of a deltoid is given by

$$t \mapsto \frac{a}{3}\left(8\cos\frac{t}{2} + 2\cos t - \cos 2t,\ -8\sin\frac{t}{2} + 2\sin t + \sin 2t\right).$$

Unlike any Reuleaux polygon, this is a *regular* curve of constant width. It is plotted in Figure 6.13.

6.8 The Support Function of an Oval

Given a simple closed curve $\mathscr{C}$, choose a point $\mathbf{o}$ inside $\mathscr{C}$. Let m denote the tangent line to $\mathscr{C}$ at a point $\mathbf{r}$ on $\mathscr{C}$. Let ℓ denote the line through $\mathbf{o}$ meeting m perpendicularly at a point $\mathbf{p}$. As $\mathbf{r}$ traces out $\mathscr{C}$, so $\mathbf{p}$ traces out out the associated pedal curve defined in Section 4.6. Take $\mathbf{o}$ to be the origin of coordinates, and let ψ be the angle between ℓ and the x-axis. Set

$$p = \text{length } \overline{\mathbf{op}};$$

the diagram is Figure 6.17 on page 178.

The point $\mathbf{r}$ is given in polar coordinates as $re^{i\theta}$, where $r =$ length $\overline{\mathbf{or}}$, and θ is the angle between $\overline{\mathbf{or}}$ and the x-axis. Then $p = r\cos(\psi - \theta)$, so that the line m is given by

$$p = r\cos(\psi - \theta) = x\cos\psi + y\sin\psi. \tag{6.17}$$

Since x, y can themselves be expressed in terms of ψ, we ultimately obtain p as a function of ψ. Conversely, let $p(\psi)$ be a given function of ψ; for each value of ψ, (6.17) defines a straight line, and so we obtain a family of lines. This family is defined by

$$F(x, y, \psi) = 0, \tag{6.18}$$

where $F(x, y, \psi) = p(\psi) - x\cos\psi - y\sin\psi$.

The discourse that follows applies in greater generality to a family of straight lines (or indeed, curves). When a given ψ is replaced by $\psi + \delta$, we obtain a new line implicitly defined by

$$F(x, y, \psi + \delta) = 0. \tag{6.19}$$

The set of points that belong to both curves satisfies

$$\frac{F(x, y, \psi + \delta) - F(x, y, \psi)}{\delta} = 0. \tag{6.20}$$

When we take the limit as δ tends to zero in (6.20), we obtain

$$\frac{\partial F(x, y, \psi)}{\partial \psi} = 0. \tag{6.21}$$

One calls the curve implicitly defined by eliminating ψ from (6.18) and (6.21) the ***envelope*** of the family of lines. In imprecise but descriptive language, we say that the envelope consists of those points which belong to each pair of infinitely near curves in the family (6.18). A similar argument shows that the evolute of a plane curve is the envelope of its normals, exhibited in Figure 4.12 on page 111.

In the case at hand, (6.18) and (6.21) become

$$\left\{ \begin{array}{rcl} x\cos\psi + y\sin\psi &=& p(\psi), \\ -x\sin\psi + y\cos\psi &=& p'(\psi). \end{array} \right. \tag{6.22}$$

Their solution (6.22) is

$$\left\{ \begin{array}{rcl} x &=& p(\psi)\cos\psi - p'(\psi)\sin\psi, \\ y &=& p(\psi)\sin\psi + p'(\psi)\cos\psi, \end{array} \right. \tag{6.23}$$

and the parameter ψ can be used to parametrize $\mathscr{C}$.

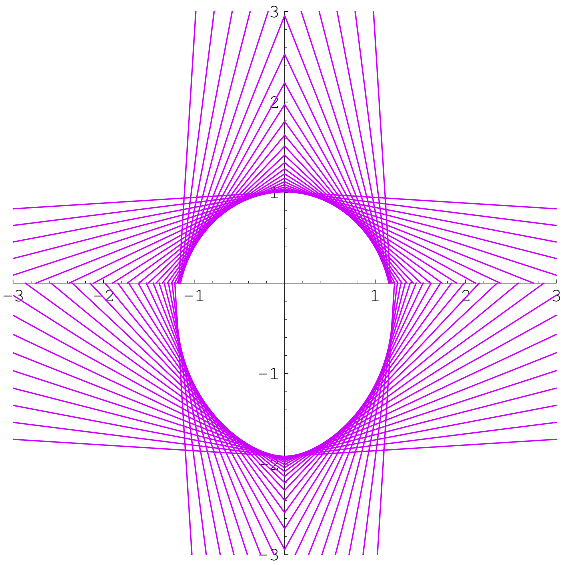

Figure 6.14: The family of straight lines for $p(\psi) = \csc(1 + c \sin\psi)$

Definition 6.30. *Let $\mathscr{C}$ be a plane curve and* $\mathbf{o}$ *a point. The* ***support function*** *of $\mathscr{C}$ with respect to* $\mathbf{o}$ *is the function p defined by* (6.17). *The* ***pedal parametrization*** *of $\mathscr{C}$ with respect to* $\mathbf{o}$ *is*

$$\mathsf{oval}[p](\psi) = \big(p(\psi)\cos\psi - p'(\psi)\sin\psi,\ p(\psi)\sin\psi + p'(\psi)\cos\psi\big). \tag{6.24}$$

This formula defines a curve for any function p. However, it will only be closed if p is periodic; this fails in the example illustrated by Figure 6.16. The curve defined by (6.24) will be an oval if and only if it is a simple closed curve and $\boldsymbol{\kappa}\mathbf{2}$ is never zero. This is certainly the case in Figure 6.15, the envelope determined by the lines in Figure 6.14.

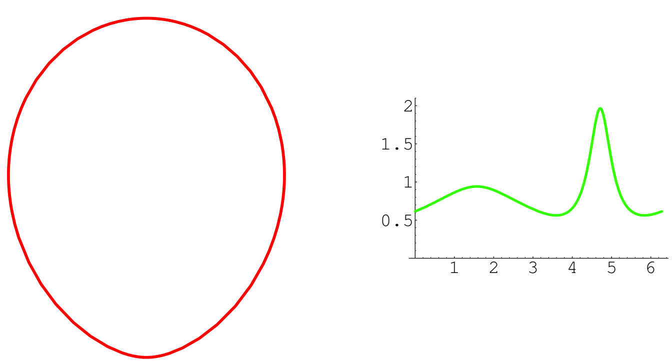

Figure 6.15: An egg and its curvature

Lemma 6.31. *Let $p\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function. The curvature of* $\mathsf{oval}[p]$ *is given by*

$$\boldsymbol{\kappa}\mathbf{2}(\psi) = \frac{1}{p(\psi) + p''(\psi)}. \tag{6.25}$$

As a consequence, $\mathsf{oval}[p]$ *is an oval if and only if it is a simple closed curve and* $p(\psi) + p''(\psi)$ *is never zero.*

Proof. Equation (6.25) is an easy calculation from (6.24) and the definition of $\boldsymbol{\kappa}\mathbf{2}$. It can also be checked by computer; see Exercise 10. ▮

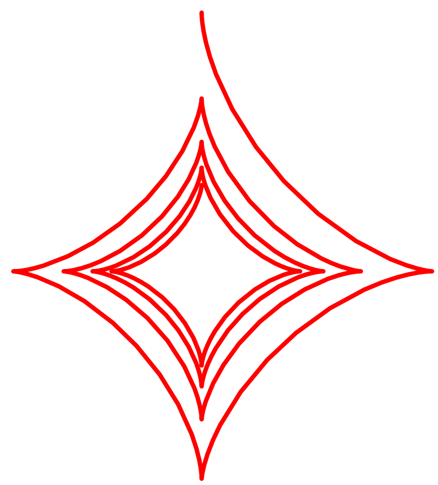

Figure 6.16: The nonclosed curve with $p(\psi) = \sin(2\psi)/\psi$

There is a formula relating the width and support functions of an oval.

Lemma 6.32. *Let p be the support function of an oval $\mathscr{C}$ with respect to some point $\mathbf{o}$ inside $\mathscr{C}$, and let μ denote the width function of $\mathscr{C}$. Then*

$$\mu(\psi) = p(\psi) + p(\psi + \pi) \tag{6.26}$$

for $0 \leqslant \psi \leqslant 2\pi$.

Proof. Let $\mathbf{p}$ be a point on $\mathscr{C}$, and let $\mathbf{p}_1 = \mathbf{p}^{\mathrm{o}}$ denote the point on $\mathscr{C}$ opposite to $\mathbf{p}$. By definition the ray $\overline{\mathbf{op}}$ from $\mathbf{o}$ to $\mathbf{p}$ meets the tangent line to $\mathscr{C}$ at $\mathbf{p}$ perpendicularly, and similarly for $\widehat{\mathbf{p}}$. Hence the rays $\overline{\mathbf{op}}$ and $\overline{\mathbf{op}_1}$ are part of a line segment ℓ that meets each of the two tangent lines perpendicularly. The length of ℓ is the width of the oval measured at $\mathbf{p}$ or at $\widehat{\mathbf{p}}$. Since the length of $\overline{\mathbf{op}}$ is $p(\psi)$ and the length of $\overline{\mathbf{op}_1}$ is $p(\psi + \pi)$, we obtain (6.26). ▮

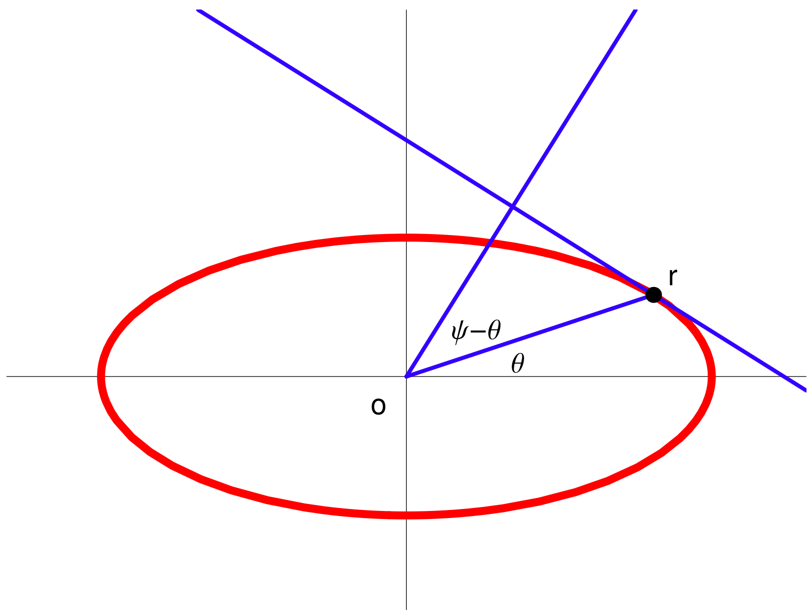

Figure 6.17: Parametrizing an oval by $p(\psi)$

6.9 Exercises

1. Consider the ***piriform*** defined by

$$\text{piriform}[a,b](t) = \big(a(1+\sin t),\ b\cos t(1+\sin t)\big).$$

Find a point where this curve fails to be regular, and plot $\text{piriform}[1,1]$.

2. Finish the proof of Lemma 6.2.

M **3.** Show that the figure eight curve

$$t \mapsto (\sin t,\ \sin t\cos t), \qquad 0 \leqslant t \leqslant 2\pi$$

(from page 44) and the lemniscate

$$t \mapsto \left(\frac{a\cos t}{1+\sin^2 t}, \frac{a\sin t\cos t}{1+\sin^2 t}\right), \qquad 0 \leqslant t \leqslant 2\pi$$

(from page 43), are both closed curves with total signed curvature and turning number equal to zero. Check the results by computer.

M **4.** Show that the limaçon

$$t \mapsto (2a\cos t + b)(\cos t,\ \sin t), \qquad 0 \leqslant t \leqslant 2\pi$$

(from page 58) has turning number equal to 2. Why is the turning number of a limaçon different from that of a figure eight curve or a lemniscate?

5. Verify that limacon$[1,1]$ is a nonsimple closed curve that has exactly two vertices, and plot its curvature.

6. Check that

$$\text{epitrochoid}[a,b,h](t) = \text{hypotrochoid}[a,-b,-h](t),$$

and explain this with reference to the definitions and Figure 6.18.

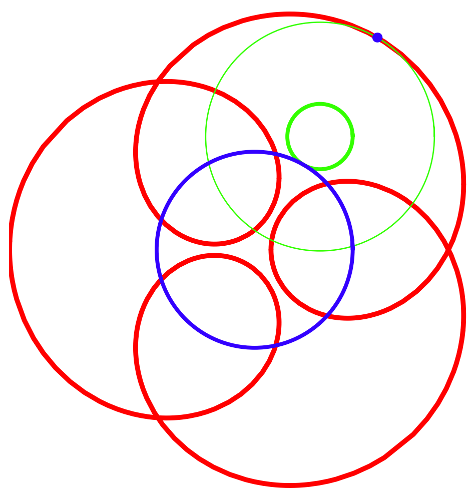

Figure 6.18: Definition of the epitrochoid

M **7.** Plot hypotrochoid$[18,2,6]$ and compute its turning number.

M **8.** Find an explicit formula for the signed curvature of a general epitrochoid.

9. Verify that a noncircular ellipse has exactly four vertices and plot its curvature.

M **10.** Prove (6.25) by computer.

11. Show that a parallel curve $\mathscr{P}$ to a closed curve $\mathscr{C}$ of constant width also has constant width, provided $\mathscr{C}$ is interior to $\mathscr{P}$.

12. Referring to Figure 6.17, suppose that $\mathbf{o}=(0,0)$ is the center of an ellipse $\mathscr{E}$ parametrized as $(a\cos t,\ b\sin t)$. Find a relationship between $\tan t$ and $\tan\psi$, and deduce that the pedal parametrization of $\mathscr{E}$ can be obtained frrom the equation

$$p=\sqrt{(a^2-b^2)\cos^2 t+b^2}$$

13. Complete the determination of C_1, C_2 in the proof of Theorem 6.28. In the same notation, verify that the spread of an oval is given by

$$\lambda(\vartheta)=-\big(U(\vartheta)-\tfrac{1}{2}U(\pi)\big)\cos\vartheta-\big(V(\vartheta)-\tfrac{1}{2}V(\pi)\big)\sin\vartheta.$$

14. The straight lines in Figure 6.13 join opposite points on $R[1,1]$, where 'opposite' now means 'half the total arc length apart'. Investigate the deltoid-shaped curve formed as the envelope of this family.