Stiefel-Whitney Classes and Spin Structures on Flat Manifolds of diagonal Type

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Abstract. xxxxxx

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INTRODUCTION

The connection between the spectrum of the Laplace operator of a compact Riemannian manifold and its geometry and topology has been a subject of study for quite some time. The original examples of isospectral but not isometric manifolds were found by Milnor – these are flat tori. Since that point, an enormous amount of effort has been put into finding manifolds which are isospectral but not isometric (“one cannot hear the shape of a drum”) and which even have different topologies.

These kind of problems have been investigated in the context of nilmanifolds, solv-manifolds and compact flat manifolds. The latter turn out to be a rich family where one can rather explicitly compute the multiplicities of eigenvalues of Laplace type operators, the real cohomology and the lengths of closed geodesics. Two Riemannian manifolds are p-isospectral if they have the same spectrum with respect to the Hodge Laplacian \( \Delta_p \) acting on p-forms.

In [2] we showed that “one cannot hear both \( \mathbb{Z}_2 \)-cohomology and 2nd Stiefel-Whitney classes”. This was achieved studying the \( \mathbb{Z}_2 \)-cohomology groups for compact flat Riemannian manifolds of diagonal type \( M_\Gamma = \Gamma \backslash \mathbb{R}^n \) by explicit computation of the differentials in the Lyndon-Hochschild-Serre (LHS) spectral sequence. Expressions for \( H^j(M_\Gamma, \mathbb{Z}_2) \), \( j = 1, 2 \) were obtained and an effective criterion for the non-vanishing of the second Stiefel-Whitney class \( w_2(M_\Gamma) \) was given (cf. Theorem 2 here).

Recall that there is a Spin structure on an oriented Riemannian manifold \( M \) if and only if \( w_2(M) = 0 \) [4].

In [3] were given necessary and sufficient conditions for the existence of Spin (and Pin\( ^\pm \)) structures on Riemannian manifolds with holonomy group \( \mathbb{Z}_2^k \). As an application, it was studied whether one can hear the property of being Spin on a compact Riemannian manifold. It was shown that the property of being Spin (resp. Pin\( ^\pm \)) cannot be heard, by exhibiting isospectral manifolds \( M_1, M_2 \) such that \( M_1 \) is Spin (resp. Pin\( ^\pm \)) but \( M_2 \) has no
Spin (resp. Pin±) structure.

In this note we relate the necessary and sufficient condition in [3] with the non-vanishing of the Stiefel-Whitney class w2(MΓ). This yields an alternate method to obtain the examples in [3].

**BIEBERBACH GROUPS**

A crystallographic group is a discrete, cocompact subgroup Γ of the isometry group of \( \mathbb{R}^n \). \( I(\mathbb{R}^n) \cong O(n) \times \mathbb{R}^n \). If Γ is also torsion-free, then Γ is a Bieberbach group.

Such Γ acts properly discontinuously and freely on \( \mathbb{R}^n \), thus \( M_\Gamma = \Gamma \backslash \mathbb{R}^n \) is a compact flat Riemannian manifold with fundamental group Γ. Any compact flat manifold arises in this way.

By Bieberbach’s first Theorem, pure translations in Γ form a normal maximal abelian subgroup of finite index, \( L_\Lambda \), \( \Lambda \) a lattice in \( \mathbb{R}^n \). The quotient \( F = \Lambda \Gamma \) gives the linear holonomy group of \( M_\Gamma \).

Any \( \gamma \in I(\mathbb{R}^n) \) can be written uniquely \( \gamma = BL_b \), where \( B \in O(n) \) and \( L_b \) is a translation by \( b \in \mathbb{R}^n \). The restriction to \( \Gamma \) of the canonical projection \( r : I(\mathbb{R}^n) \to O(n) \), \( r(BL_b) = B \), is a homomorphism with kernel \( \Lambda \simeq L_\Lambda \), and \( r(\Gamma) \cong F \) is a finite subgroup of \( O(n) \) called the point group of \( \Gamma \). Algebraically, \( \Gamma \) is an extension of \( F \) by \( \Lambda \), i.e., there is an exact sequence \( 0 \to \Lambda \to \Gamma \to F \to 1 \).

Since \( L_\Lambda \) is a normal subgroup of \( \Gamma \), and \( (BL_b)L_\lambda (BL_b)^{-1} = L_{B\lambda} \), then \( \Lambda \) is \( B \)-stable for any \( BL_b \in \Gamma \). Thus conjugation by \( BL_b \) induces an action of \( F \) on \( \Lambda \) which is given by \( \lambda \in \Lambda \mapsto B\lambda \) and is called the holonomy representation. If the holonomy representation diagonalizes the corresponding manifolds are called compact flat manifolds of diagonal type.

**Example 1 (Hantzsche-Wendt (HW) 3-manifold or didicosm)** \( M_\Gamma \) is the flat manifold of dimension \( n = 3 \), whose holonomy group of \( \Gamma = \mathbb{Z}_2^3 \) is generated by \( B_1 = \text{diag}(1, -1, -1) \), \( B_2 = \text{diag}(-1, 1, -1) \), \( b_1 = \frac{e_1 + e_3}{2} \), \( b_2 = \frac{e_1 + e_2}{2} \); i.e., \( \Gamma = \langle B_1L_{b_1}, B_2L_{b_2}; L_{\mathbb{Z}_2^3} \rangle \).

A standard way to represent Bieberbach groups of diagonal type is by using the column notation, i.e., writing the diagonal matrices in columns and as a subscript, the corresponding translation vector modulo the lattice.

**Example 2** In column notation the HW 3-manifold is represented as follows:

\[
\begin{array}{c|c}
B_1 & B_2 \\
\hline
1 & -1 \\
-1 & 1 \\
-1 & -1 \\
\end{array}
\quad \text{or} \quad
\begin{array}{c|c|c}
B_1 & B_2 & B_1B_2 \\
\hline
1 & -1 & -1 \\
-1 & 1 & -1 \\
-1 & -1 & 1 \\
\end{array}
\]

We denote by \( \gamma_j \) the row \( j \) element \( \pm 1 \), in \( \gamma = BL_b \).
Observe that $M_\Gamma$ is orientable if and only if $\det B = 1$ for every $BL_b \in \Gamma$. For a flat manifold of diagonal type, in column notation this means that each column has an even number of $-1$’s.

A group extension is determined by a cohomology class $\beta \in H^2(F, \Lambda)$. This class determines a $\mathbb{Z}_2$-cohomology class $\bar{\beta} \in H^2(F, \Lambda \otimes \mathbb{Z}_2)$.

Suppose $\Gamma$ has diagonal holonomy $\mathbb{Z}_2^k = \langle B_1, \ldots, B_k \rangle$, then $\bar{\beta} \in H^2(\mathbb{Z}_2^k, \Lambda \otimes \mathbb{Z}_2) \cong (H^2(\mathbb{Z}_2^k, \mathbb{Z}_2))^n$ $(n = \dim M_\Gamma)$. Now $H^*(\mathbb{Z}_2^k, \mathbb{Z}_2)$ is the polynomial algebra $\mathbb{Z}_2[x_1, \ldots, x_k]$ with generators $x_1, \ldots, x_k$ in degree one. Thus, the components $\bar{\beta}_i$ of $\bar{\beta}$ are degree 2 homogeneous polynomials: the $\mathbb{Z}_2$-class polynomials which are explicitly given by the formula [2, Proposition 1.3]

$$\bar{\beta}_i = \sum_{i : B_i e_\ell = e_\ell} x_i^2 + \sum_{i : b_{i \ell} = \frac{1}{2}} \sum_{j \neq i} x_i x_j, \quad b_{i \ell} = \frac{1}{2}$$

where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$.

It turns out that the $\mathbb{Z}_2$-class polynomials determine the differentials in the LHS spectral sequence [2]. In particular $H^1(\Lambda, \mathbb{Z}_2) \xrightarrow{d^0_{\mathbb{Z}_2}} H^2(\mathbb{Z}_2^k, \mathbb{Z}_2)$ is given by $d^0_{\mathbb{Z}_2}(e^i) = \bar{\beta}_i$, where $(e^i)$ is a basis of $H^1(\Lambda, \mathbb{Z}_2) \cong (\mathbb{Z}_2^n)^*$. In particular, the image of $d^0_{\mathbb{Z}_2}$ is given by the linear span of the $\mathbb{Z}_2$-class polynomials and therefore (the coefficients being in $\mathbb{Z}_2$) by the sums of the $\mathbb{Z}_2$-class polynomials.

**Remark 1** The expression of the $\mathbb{Z}_2$-class polynomial can be described in more graphic terms. In column notation, the $\ell$-component $\bar{\beta}_\ell$ is a polynomial obtained from the $\ell$-th row of the diagram, by including the term $x_i^2$ if the entry $(\ell, i)$ is $1_\frac{1}{2}$, and including the term $x_i x_j$, for $i \neq j$, if the entries $(\ell, i)$, $(\ell, j)$, where $\ell$ is either 0 or $\frac{1}{2}$. Visually, we have

<table>
<thead>
<tr>
<th>$B_i$</th>
<th>$\mathbb{Z}_2$-class polynomial</th>
<th>$B_i$</th>
<th>$B_j$</th>
<th>$\mathbb{Z}_2$-class polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1_\frac{1}{2}$</td>
<td>$x_i^2$</td>
<td>$1_\frac{1}{2}$</td>
<td>$-1_u$</td>
<td>$x_i^2 + x_i x_j$</td>
</tr>
<tr>
<td>$-1_u$</td>
<td>$\frac{1}{2} - u$</td>
<td>$-1_u$</td>
<td>$\frac{1}{2} - u$</td>
<td>$x_i x_j$</td>
</tr>
</tbody>
</table>

A basic observation that will be important in the sequel is the following. A quadratic term $x_i^2$ in the $\mathbb{Z}_2$-class polynomial $\bar{\beta}_i$ arises exclusively from elements $\gamma^i$ of the kind $1_\frac{1}{2}$.

**Example 3** ($\mathbb{Z}_2$-class polynomial for the HW 3-manifold) If we apply this rule in the case of the HW 3-manifold (see Example 1), the $\mathbb{Z}_2$-class polynomials are $\bar{\beta}_1 = x_1^2 + x_1 x_2$, $\bar{\beta}_2 = x_1^2 + x_1 x_2$, $\bar{\beta}_3 = x_1 x_2$.

**STIEFEL-WHITNEY CLASSES**

For a compact flat manifold $M_\Gamma$ of diagonal type consider the following composition of maps at the level for the corresponding Bieberbach group $\Gamma$

$$\Gamma \xrightarrow{\gamma} F \cong \mathbb{Z}_2^k \xrightarrow{i} D(n) \cong \mathbb{Z}_2^n \rightarrow O(n)$$
(here, $D(n) \cong \mathbb{Z}_2^n$ denotes the diagonal matrices in $O(n)$.) The above map induces a map of $M$ into $BO(n)$ which is the classifying map for $TM$.

Let $x_1, \ldots, x_k$ be a basis of $H^1(\mathbb{Z}_2^k, \mathbb{Z}_2)$ and let $x_1', \ldots, x_n'$ be the standard basis of $H^1(D(n), \mathbb{Z}_2)$. The classes $\omega_\ell = i^*(x_\ell')$ are called the 2-weights of the map $i$. Observe that $\omega_\ell = \sum_m a_{\ell m}x_m$ where $a_{\ell m} = 1$ if the entry $(\ell, i)$ is $-1$ and it is zero if it is $1$.

**Example 4** In the case of the HW 3 manifold (cf. Example 1) the 2-weights are

\[ \omega_1 = x_2, \quad \omega_2 = x_1, \quad \omega_3 = x_1 + x_2. \]

Let $\sigma_j(\omega_1, \ldots, \omega_n)$ be the $j$-th elementary symmetric function in $\omega_1, \ldots, \omega_n$.

**Proposition 1** [2] Given the canonical projection $\Gamma \rightarrow F = \mathbb{Z}_2^k$, we have

\[ w_j(M) = r^* \sigma_j(\omega_1, \ldots, \omega_n). \]

By the LHS Exact Sequence

\[ \cdots \rightarrow H^1(\Lambda, \mathbb{Z}_2) \xrightarrow{d_0^1} H^2(F, \mathbb{Z}_2) \xrightarrow{r^*} H^2(\Gamma, \mathbb{Z}_2) \]

So, $\ker r^* = \text{Im} \ d_2^{0,1}$ is spanned by the $\mathbb{Z}_2$-class polynomials $\tilde{\beta}_\ell (\ell = 1, \ldots, n)$ and we have

**Theorem 2** [2] Let $M_\Gamma$ be an $n$-dimensional compact flat manifold with diagonal holonomy $\mathbb{Z}_2^k$. Then $w_2 \neq 0 \iff \sigma_2(\omega_1, \ldots, \omega_n)$ is not a sum of $\mathbb{Z}_2$-class polynomials.

**CRITERION ON EXISTENCE OF SPIN STRUCTURES**

We now relate the criterion in [3] for the non-existence of Spin structures with the non-vanishing of the Stiefel-Whitney class $w_2(M_\Gamma)$ expressed in terms of Theorem 2.

We begin with the following

**Definition 1** Let $\mathcal{P}\{1, \ldots, n\}$ be the power set of $\{1, \ldots, n\}$ and define $c : \Gamma \rightarrow \mathcal{P}\{1, \ldots, n\}$ as the map $\gamma \in \Gamma \mapsto \{i_1, \ldots, i_l\}$ if $\gamma^j = 1_\gamma$.

For $\gamma = BL_b$ define $n_\gamma = n_B$ to be the number of 1’s in $B$. Thus $n - n_B$ is the number of $-1$’s in $B$.

The condition for non existence of Spin structures in [3] can be restated (in additive notation) in case of diagonal holonomy as follows

**Proposition 3** A compact orientable flat manifold of diagonal type $M_\Gamma$ has no Spin structure if and only if for any $g : \{1, \ldots, n\} \rightarrow \{0, 1\}$ there exists a $\gamma \in \Gamma$ such that

\[ \sum_{i \in c(\gamma)} g(i) \equiv \frac{n - n_\gamma}{2} \pmod{2}. \]
We now prove the following result

**Theorem 4** For any \( g : \{1, \ldots, n\} \to \{0, 1\} \) there exists a \( \gamma \in \Gamma \) such that

\[
\sum_{i \in c(\gamma)} g(i) \not\equiv \frac{n-n\gamma}{2} \pmod{2}.
\]

if and only if \( w_2(M_{\Gamma}) \neq 0 \).

**Proof.**  
\( \Rightarrow \) Take \( \gamma := \gamma_j \) as one of the generators of \( \Gamma \). Hence

\[
\sum_{i \in c(\gamma_j)} g(i) \not\equiv \frac{n-n\gamma_j}{2} \pmod{2}.
\]

and therefore the coefficient of \( x_j^2 \) in \( \sum_{i=1}^n g(i)\beta_i \) is different from the coefficient of \( x_j^2 \) in \( \sigma_2 := \sigma_2(\omega_1, \ldots, \omega_n) \). Hence \( \sigma_2 \neq \sum_{i=1}^n g(i)\beta_i \) for any \( g \), i.e., \( \sigma_2 \) is not a sum of \( \mathbb{Z}_2 \)-class polynomials. Thus \( w_2 \neq 0 \).  

\( \Leftarrow \) Suppose \( w_2 \neq 0 \). By Theorem 2, this is equivalent to the fact that any linear combination (actually sum, since coefficients are in \( \mathbb{Z}_2 \)) of \( \mathbb{Z}_2 \)-class polynomials does not yield \( \sigma_2 \). This is in turn equivalent that, for any \( g : \{1, \ldots, n\} \to \{0, 1\} \), \( \sum_{i=1}^n g(i)\beta_i \neq \sigma_2 \) or that, for any \( g \), there is a monomial \( m \) in \( \sum_{i=1}^n g(i)\beta_i \) such that

the coefficient of \( m \) in \( \sum_{i=1}^n g(i)\beta_i \neq \) the coefficient of \( m \) in \( \sigma_2 \). \hspace{1cm} (1)

Now we have two cases:

1. \( m \) is quadratic, i.e., \( m = x_i^2 \) for some \( j \);
2. there is no quadratic such \( m \), i.e., \( m = x_ix_k \) for some \( j,k \) (\( j \neq k \)) but all the coefficients for \( x_i^2 \) for any \( i \) coincide.

In both cases we will show that there exists a \( \gamma \in \Gamma \) such that

\[
\sum_{i \in c(\gamma)} g(i) \not\equiv \frac{n-n\gamma}{2} \pmod{2}.
\]

In case 1 we will have that \( \gamma = \gamma_j \) (a generator), in the case 2 that \( \gamma \) is the product of two generators \( \gamma_j \gamma_k \).

**Case 1:** the coefficient of \( x_j^2 \) in \( \sum_{i=1}^n g(i)\beta_i \) is given by \( \sum_{i \in c(\gamma)} g(i) \pmod{2} \) (this is because of how the \( \beta_i \) come out; cf. Remark 1). Furthermore, the coefficient of \( x_j^2 \) in \( \sigma_2 \) is given by \( \left(\frac{n-n\gamma_j}{2}\right) \pmod{2} \), or, simpler, by \( \frac{n-n\gamma_j}{2} \pmod{2} \) (for this computation, we are using that \( n-n\gamma_j \) is even, i.e., that \( M_{\Gamma} \) is orientable). Thus, by (1), we have

\[
\sum_{i \in c(\gamma)} g(i) \not\equiv \frac{n-n\gamma_j}{2} \pmod{2}.
\]
Case 2: if we denote by $\text{coeff}/g$ the coefficient is the linear combination $\sum_{i=1}^{n} g(i)$, we know that

- $\text{coeff}/g$ of $x_i^2$ in $\ker r^*$ \(\equiv\) $\text{coeff}/g$ of $x_i^2$ in $\sigma_2(\mod 2)$,
- $\text{coeff}/g$ of $x_k^2$ in $\ker r^*$ \(\equiv\) $\text{coeff}/g$ of $x_k^2$ in $\sigma_2(\mod 2)$,
- $\text{coeff}/g$ of $x_jx_k$ in $\ker r^*$ \(\not\equiv\) $\text{coeff}/g$ of $x_jx_k$ in $\sigma_2(\mod 2)$.

So

$$\sum \text{coeff}/g \text{ of } x_j^2, x_k^2, x_jx_k \in \ker r^* \not\equiv \sum \text{coeff}/g \text{ of } x_j^2, x_k^2, x_jx_k \in \sigma_2(\mod 2). \tag{2}$$

Next we compute both sides in this inequality.

**Left-hand side:** according to the way one computes the $\mathbb{Z}_2$-class polynomials, the only lines that contribute are those of the form $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ which are exactly the entries with $1 \gamma \in \gamma = \gamma_j \gamma_k$. Hence the left-hand side is $\sum_{i \in c(\gamma)} g(i)(\mod 2)$.

**Right-hand side:** Let $a, b, c$ denote respectively the lines of the form

- $-1 \quad -1 : a$
- $-1 \quad +1 : b$
- $+1 \quad -1 : c$

Then

- $\text{coeff of } x_i^2 \in \sigma_2 = (\frac{a}{2}) + (\frac{b}{2}) + ab$,
- $\text{coeff of } x_k^2 \in \sigma_2 = (\frac{a}{2}) + (\frac{c}{2}) + ac$,
- $\text{coeff of } x_jx_k \in \sigma_2 = ab + ac + bc$.

So

$$\sum \text{coeff of } x_j^2, x_k^2, x_jx_k \in \sigma_2 \equiv (\mod 2) bc + \frac{b}{2} + \frac{c}{2} = \left( \frac{b+c}{2} \right) = \left( \frac{n-n\gamma}{2} \right).$$

Thus, by (2), $\sum_{i \in c(\gamma)} g(i) \not\equiv (\frac{n-n\gamma}{2}) (\mod 2)$. \(\square\)

**REFERENCES**