SYMMETRIC WEIGHTS AND S-REPRESENTATIONS

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Abstract

We study irreducible representations of compact Lie groups relating an algebraic condition (the highest weight \( \lambda \) is "symmetric", i.e., in any simple factor all non zero \( \langle \lambda, \alpha \rangle \) are equal, for any positive root \( \alpha \) and any invariant inner product) with a geometric one (for all orbits, the \( d \)-th osculating space coincides with the representation space).

We prove that, if \( d = 2 \) and \( \lambda \) is symmetric, the irreducible representation with highest weight \( \lambda \) corresponds to the isotropy representation of a symmetric space.

1. Introduction

Let \( K \) be a compact Lie group and \( \phi \) a faithful irreducible orthogonal representation. Our aim is to investigate the interplay between algebraic properties of the weight system of \( \phi \) and geometric properties of the representation \( \phi \).

Among orthogonal representations, a crucial rôle in submanifold geometry is played by the isotropy representations of symmetric spaces, called s-representations. Indeed the principal orbits of s-representations are isoparametric and the singular ones are their focal manifolds. Moreover all orbits of s-representations are taut [2].

The s-representations are strictly related to another class of orthogonal representations whose definition is geometrically more appealing: the polar representations. A representation of a compact Lie group \( K \) on vector space \( V \) is polar if there is a linear subspace \( \Sigma \subset V \) that meets all orbits of \( K \) and every time it meets an orbit of \( K \), it meets it perpendicularly. It is not difficult to see that any s-representation is polar. Moreover it is still true that any orbit of a polar representation is taut, as it follows from results of Conlon [4] together with ones of Bott and Samelson [2].

On the other hand, Dadok [6] classified all irreducible polar representations and observed that some of them are s-representations and that, those that are not, have the same orbits as s-representations. For his classification, Dadok associated to any irreducible representation with highest weight \( \lambda \), an integer \( k(\lambda) \).
He proved that for a polar representation one has the upper bound $k(\lambda) \leq 4$, if it is of real type, and $k(\lambda) \leq 2$, otherwise. This result was crucial in his proof since it reduced considerably the list of possible polar representations.

We will give a geometric interpretation of this upper bound on $k(\lambda)$ for polar representations. This is done in Section 2, where we consider, like in [5] the class $(\mathbb{O}_2)$ of orthogonal representations. In general, $\phi$ belongs to $(\mathbb{O}_d)$ if the $d$-th osculating space coincides with $V$. Any irreducible polar representation belongs to $(\mathbb{O}_2)$ and we prove that the above upper bound for $k(\lambda)$ holds, more generally, for irreducible representations of class $(\mathbb{O}_2)$ (Theorem 2.1).

The next part of our work starts from the observation that, up to a few exceptions ($SU^*(2n)/Sp(n)$ and $E_6/F_4$), the irreducible polar representations that are $s$-representations (and not just orbit equivalent to them) and not transitive on the unit sphere are those for which $k(\lambda)$ assumes its maximal value mentioned above (i.e., $k(\lambda) = 4$ if $\phi$ is of real type and $k(\lambda) = 2$ otherwise) and whose highest weight is symmetric, i.e., all nonzero $\langle \lambda, \alpha \rangle$ are equal for any positive root $\alpha$ chosen in any simple factor of $\mathfrak{f}$, where $\langle , \rangle$ is any $\mathfrak{f}$-invariant inner product (cf. Theorem 9 (ii) and Theorem 10 (ii) in [6]).

Rather than giving a new proof of Dadok's Theorem (this was done by Eschenburg and Heintze in [8], using submanifold geometry) we aim to study the interplay between $s$-representations and representations with symmetric highest weight $\lambda$ and for which $k(\lambda)$ assumes the maximal value allowed for the class $(\mathbb{O}_2)$. Our main result on one hand generalizes to some extent Theorems 9 and 10 in [6], since we do not assume the representation to be polar; on the other, it gives an alternative proof of them.

**Theorem 1.1.** Let $\phi_\lambda : K \to O(V)$ be a faithful irreducible complex representation of a semisimple, compact, connected Lie group $K$ with highest weight $\lambda$, and let $\langle , \rangle$ be a $\mathfrak{f}$-invariant inner product on $\mathfrak{f}$.

(a) If $\phi_\lambda$ is of real type, $k(\lambda) = 4$ and $\lambda$ is symmetric then $\phi_\lambda^R$ is the isotropy representation of a compact, simply connected, irreducible symmetric space.

(b) If $\phi_\lambda$ is of complex type, $k(\lambda) = 2$ and $\lambda$ is symmetric then $(K \cdot U(1), [\phi_\lambda \otimes e^{i\theta}]_R)$ is the isotropy representation of a compact, simply connected, irreducible hermitian symmetric space.

(c) If $\phi_\lambda$ is of symplectic type, $k(\lambda) = 3$ (thus is this case $\phi_\lambda$ cannot belong to $(\mathbb{O}_3)$) and $\lambda$ is symmetric then $(K \cdot Sp(1), \phi_\lambda \otimes v_2)$ is the isotropy representation of a compact, simply connected, irreducible quaternionic symmetric space.

Note that in case (c) $\phi_\lambda \otimes v_2$ is of real type, its highest weight $\lambda'$ has $k(\lambda') = 4$. Thus (c) is a special case of (a).

For the proof we state some properties of irreducible representations whose highest weight is symmetric and $k(\lambda)$ is maximal for $(\mathbb{O}_2)$ (Lemma 3.1 and 3.2). These properties generalize to the ones for which $k(\lambda)$ is maximal for $(\mathbb{O}_d)$. What turns out to be crucial in the case of class $(\mathbb{O}_2)$ is that in this case, up to a few special cases, $\lambda$ is a sum of minuscule weights, each in any simple factor of $K$.

We wish to thank McKenzie Wang for his helpful comments.
2. Osculating spaces and weights

In this Section we want to give a geometric proof of the upper bound for $k(\lambda)$. To this purpose, we first recall some facts from representation theory, then we give the definition of $k(\lambda)$ and finally we relate $k(\lambda)$ with the decomposition of the representation space into osculating spaces.

Let $\phi$ be an irreducible representation of a compact Lie group $K$ on a complex vector space $V$. $V$ will be always endowed with a $K$-invariant inner product (which is uniquely defined up to a constant factor). Let $\mathfrak{t}^C$ be the complexification of the Lie algebra $\mathfrak{t}$ of $K$ and $\phi^C$ the corresponding irreducible representation of $\mathfrak{t}^C$ on the complex vector space $V$.

One says that $\phi$ is of real type if $\phi^C$ comes from a representation of $\mathfrak{t}^C$ on a real vector space $W$ by extension of scalars (i.e., $V = W \otimes_R \mathbb{C}$). This is equivalent to the existence of an invariant real structure on $V$, i.e., a conjugate linear endomorphism $\mathcal{J}$ of $V$ such that $\mathcal{J}^2 = id$. The representation $\phi$ is of symplectic type if it comes from a quaternionic representation by restriction of scalars, or equivalently if there exists an invariant symplectic structure, i.e., a conjugate linear endomorphism of $V$ whose square is minus the identity. One says that a representation is of complex type if it is neither real nor symplectic. Note that real and symplectic representations share the property that there exists a non degenerate invariant bilinear form on $V$. For this reason they are also called self dual. On the other hand, an irreducible representation of a complex Lie algebra $\mathfrak{t}^C$ on a real vector space is of complex or symplectic type if it comes from a complex representation by restriction of scalars. Otherwise it is of real type.

If $\phi: K \to O(V)$ is of real type we will consider the orthogonal representation $\phi^R$ on the real part $V^R$ of $V$, i.e. the $+1$-eigenspace of $\mathcal{J}$. In the complex or symplectic case we will regard $V$ as a real vector space (and when we want to stress the difference between regarding $V$ as a complex and a real vector space we will write $[\phi]^R$ in the latter case).

For representations of real type the following lemma [5] describes the real part of the representation space explicitly

**Lemma 2.1.** Let $\phi$ be an irreducible representation of real type of highest weight $\lambda$, $\{\rho\}$ the set of its positive weights and let $\{x_{\rho,1}\}$ be a union over unitary bases of the weight spaces of the positive weights. Then

$$\{v_{\rho,1} = x_{\rho,1} + \mathcal{J}x_{\rho,1}, w_{\rho,1} = i(x_{\rho,1} - \mathcal{J}x_{\rho,1})\}$$

is a basis of $V^R$, the $+1$-eigenspace of $\mathcal{J}$. Moreover $X_\alpha \mathcal{J} = -\mathcal{J}X_\alpha$ and

$$\mathcal{J}v_\lambda = v_{-\lambda}.$$

Recall that $K$ is finitely covered by a compact Lie group $\tilde{K} = K_1 \times \cdots \times K_l \times T^m$, where any $K_i$ is simple and $T^m$ is an $m$-dimensional torus. $\phi$ induces a representation $\tilde{\phi}$ of $\tilde{K}$ having the same orbits as $\phi$. Thus without loss of generality we will assume that $K = K_1 \times \cdots \times K_l \times T^m$. 
If $\phi$ is of real type then $T^m$ lies in the kernel of $\phi$. Thus, if $\phi$ is effective, then $K = K_1 \times \cdots \times K_I$.

If $\phi$ is of complex or symplectic type, then $\phi|_{T^m}$ has a one dimensional kernel. Hence, if $\phi$ is effective, then $K = K \times \cdots \times K \times T^1$. In this case $\phi$ is the external tensor product of a representation $\phi$ of the semisimple $K = K_1 \times \cdots \times K_I$ and a one dimensional representation of $T^1$ given by multiplication by $e^{i\theta}$ (which we will denote by $e^{i\theta}$). Hence $\phi = \bar{\phi} \otimes e^{i\theta}$ and, if $\bar{\phi}$ (considered as a real representation) is irreducible, $\phi$ is also.

Let $t$ be a Cartan subalgebra of $tC$ and denote by $\langle \cdot, \cdot \rangle$ an invariant inner product of $t$ (note that, on each simple factor of $t$, by Schur’s Lemma, it coincides up to a constant with the negative of the Killing form of $K$). Let $\Sigma = \{\alpha\}$ be the set of roots of $tC$ with respect to $t$ and the coroot $H_\alpha$ be given by $\langle i/H_\alpha, i/\gamma \rangle = \alpha(H)$, for any $H \in t$.

Recall that there exists a basis of $tC$, $\{H_\alpha, X_\alpha, X_{-\alpha}\}$, with

$$\langle X_\alpha, X_{-\alpha} \rangle = 1 \quad \text{and} \quad [X_\alpha, X_{-\alpha}] = H_\alpha.$$  

We will call such a basis a Chevalley basis.

One can obtain a description of the Lie algebra $t$ of the compact Lie group $K$ in terms of root vectors. This can be done as follows. Let $t_0$ denote the real subspace of $t$ consisting of the real linear combinations of the coroots $H_\alpha$. The Lie algebra $t$ of $K$ is then spanned by $it_0$, $X_\alpha - X_{-\alpha}$ and $i(X_\alpha + X_{-\alpha})$, where $X_\alpha \in tC$ is a (suitable) root vector (see, e.g. [10]).

Suppose now that $\phi$ has highest weight $\lambda$. In this case we will put an index $\lambda$, writing $\phi_\lambda$ and $V_\lambda$ and often will denote the representation as well as the representation space by $V_\lambda$. Moreover we will often write $X \cdot v$ instead of $\phi_\lambda(X)v$, for $X \in t$, $v \in V_\lambda$.

Let $v_\lambda$ be a highest weight vector of $\phi_\lambda$ and let $U(tC)$ denote the universal enveloping algebra of $tC$. Recall that

$$V_\lambda = U(tC) \cdot v_\lambda = U(n^-) \cdot v_\lambda,$$

where $n^-$ is the (nilpotent) subalgebra of $tC$ generated by $X_{-\gamma}, \gamma \in \Sigma^+$ (see, e.g. [10]).

We now come to the definition of $k(\lambda)$. Dadok [6, Proposition 7] proved that there exists a system $\mathcal{O} = \{\beta_1, \ldots, \beta_I\}$ of strongly orthogonal positive roots such that $s_0 = s_{\beta_1} \cdot s_{\beta_2} \cdots s_{\beta_I}$ is the Weyl group element mapping the positive Weyl chamber into its negative.

Note that $s_0(\lambda)$ is the smallest weight [1, Remark 2, p. 127]. Moreover if $\phi_\lambda$ is self dual (i.e. either real or symplectic) then $s_0\lambda = -\lambda$ [1, Chapitre VIII, Proposition 12, p. 132].

Let $v_{s_0(\lambda)}$ be a weight vector relative to $s_0(\lambda)$.

**Definition.** $k(\lambda)$ is the smallest integer such that $v_{s_0(\lambda)} \in U^{k(\lambda)}(tC) \cdot v_\lambda$ and $v_{s_0(\lambda)} \notin U^r(tC) \cdot v_\lambda$ for any $r < k(\lambda)$.

The system $\mathcal{O}$ of strongly orthogonal roots can be used to decide whether $\phi_\lambda$ is of real, symplectic or complex type. Moreover one can give a formula for $k(\lambda)$. Namely
Proposition 2.1.\ (i) \(\phi_\lambda\) is complex if and only if \(\lambda \notin \text{span}_R\{\beta_1, \ldots, \beta_l\}\),
(ii) \(\phi_\lambda\) is of real (resp. symplectic) type, if and only if \(\lambda \in \text{span}_R\{\beta_1, \ldots, \beta_l\}\) and \(k(\lambda)\) is an even (resp. odd) integer.

Moreover

\[
\begin{align*}
k(\lambda) &= \sum_{i=1}^l \frac{\langle \beta_i, \lambda - s_0\lambda \rangle}{\langle \beta_i, \beta_i \rangle},
\end{align*}
\]

and, in case \(\phi_\lambda\) is real or symplectic,

\[
\begin{align*}
k(\lambda) &= \sum_{i=1}^l 2 \frac{\langle \beta_i, \lambda \rangle}{\langle \beta_i, \beta_i \rangle},
\end{align*}
\]

where \(\langle , \rangle\) is any \(t\)-invariant inner product on \(t\).

Proof. The proof of the property that \(\phi_\lambda\) is complex if and only if \(\lambda\) does not belong to \(\text{span}_R\{\beta_1, \ldots, \beta_l\}\) and real or symplectic elsewhere, can be found in [6, p. 131] (this is actually in the same vein as [1, Chapitre VIII, Proposition 12, p. 132] or [13, p. 142]).

We now prove (1) and (2). Since \(s_0 = s_\beta_1 \cdot s_\beta_2 \cdots s_\beta_l\), we have that \(s_0\lambda = \lambda - \sum b_i \beta_i\). Thus \(v_{s_0\lambda} \in U \sum h_i(t^c) \cdot v_{\lambda}\) and that \(v_{s_0\lambda} \notin U^r(t^c) \cdot v_{\lambda}\) for any \(r < \sum b_i\). Moreover

\[
\begin{align*}
b_i &= \frac{\langle \beta_i, \lambda - s_0\lambda \rangle}{\langle \beta_i, \beta_i \rangle},
\end{align*}
\]

thus by definition

\[
\begin{align*}
k(\lambda) &= \sum_{i=1}^l \frac{\langle \beta_i, \lambda - s_0\lambda \rangle}{\langle \beta_i, \beta_i \rangle},
\end{align*}
\]

and if \(\lambda\) is self dual (2) follows from the fact that \(s_0\lambda = -\lambda\). \q.e.d.

Next we recall some notions of submanifold geometry. Let \(M\) be a submanifold of \(R^n\). The \(d\)-th osculating space of \(M\) at \(p\) is the space \(O_p^d(M)\) spanned by the first \(d\) derivatives in \(0\) of curves \(\gamma : (-\epsilon, \epsilon) \to M\) with \(\gamma(0) = p\). Note that \(O_p^1(M) = T_pM\). Now let \(\rho : K \to O(n)\) be a representation that we assume to be irreducible and let \(M\) be an orbit of \(K\). Since \(\rho\) is irreducible, for any \(p \in M\) there is a natural number \(h\), called the degree of the orbit, such that \(O_p^h(M) = R^n\). Remark that \(O_{p(g)}^h(M) = \rho(g)O_p^h(M),\) hence if \(O_p^h(M) = R^n\) for some \(p \in M\), then \(O_q^h(M) = R^n\) for all \(q \in M\). Notice also that there is a natural number \(d\) such that \(O_p^d(M) = R^n\) \((p \in M)\) for all orbits \(M\) of \(\rho\), and let \(d(\rho)\) denote the smallest of such numbers \(d\). In other words \(d(\rho)\) is the smallest number such that all orbits have degree \(d(\rho)\).

We will denote by \((O_d)\) the class of irreducible orthogonal representations \(\rho\) such that \(d(\rho) = d\). Notice that the smallest \(d\) for which \((O_d)\) is nonempty is
From the point of view of submanifold geometry the complexity of the orbits of \( p \) grows as \( d(p) \) gets larger.

The classes of representations we have mentioned in the Introduction (polar, \( \mathcal{S} \)-representations, with all orbits taut) all belong to \((\mathcal{C}_2)\) if we restrict ourselves to irreducible representations. To see this one has to use the following result of Kuiper [12, Théorème 2]. \textit{Let} \( M \) \textit{be a taut submanifold in} \( \mathbb{R}^n \) \textit{that is full in the sense that it is not contained in any hyperplane. Then there is a point} \( p \in M \) \textit{such that} \( \mathcal{O}^2_p(M) = \mathbb{R}^n \). This is for example true for every point \( p \in M \) that is the maximum of a distance function. Thus, in particular, we get that any irreducible polar representation belongs to \((\mathcal{C}_2)\).

Recall that a properly embedded submanifold \( M \) of \( \mathbb{R}^n \) is said to be \textit{taut} if for almost all \( x \in \mathbb{R}^n \) the distance function \( d_x : M \to \mathbb{R} ; \ p \to d(x, p)^2 \) is a \( \mathbb{Z}_2 \) perfect Morse function (i.e., the Morse inequalities with respect to \( \mathbb{Z}_2 \) are equalities).

We now consider an irreducible representation \( \phi_\lambda : K \to O(V) \). If \( \phi_\lambda \) is of real type we will say that it belongs to \((\mathcal{C}_2)\) if \( \phi_\lambda^R \) does.

The definition of osculating space and a computation shows the following

\textbf{Lemma 2.2.} Let \( \lambda \) be a highest weight, and \( v_\lambda \) a highest weight vector. Suppose \( \phi_\lambda \) is of class \((\mathcal{C}_2)\).

(i) If \( V_\lambda \) is of real type, then \( U^2(t^C) \cdot v_\lambda + U^2(t^C) \cdot v_{-\lambda} = V_\lambda \).

(ii) If \( V_\lambda \) is of complex or symplectic type, then \( U^2(t^C) \cdot v_\lambda = V_\lambda \).

The proof can be found in [5, Lemma 3]. For representations of real type, one needs Lemma 2.1.

More in general, if \( \phi_\lambda \) is of class \((\mathcal{C}_d)\), then \( U^d(t^C) \cdot v_\lambda + U^d(t^C) \cdot v_{-\lambda} = V_\lambda \), if \( V_\lambda \) is of real type, and \( U^d(t^C) \cdot v_\lambda = V_\lambda \), if \( V_\lambda \) is of complex or symplectic type.

Next we use Lemma 2.2 to show the following

\textbf{Theorem 2.1.} Let \( \phi_\lambda \) be an irreducible faithful orthogonal representation belonging to \((\mathcal{C}_2)\) (e.g. an irreducible polar representation). Then

(i) if \( \phi_\lambda \) is of real type, \( k(\lambda) \leq 4 \);

(ii) if \( \phi_\lambda \) is of complex or symplectic type, \( k(\lambda) \leq 2 \).

\textbf{Proof.} First we give a geometric interpretation of the property expressed by the Theorem. Recall that, if \( \alpha \) is a root, \( \rho \) is a weight, \( X \) belongs to the root space relative to \( \alpha \) and \( v_\rho \) is a weight vector relative to \( \rho \), \( X \cdot v_\rho \) belongs to the weight space \( V_{\alpha+\rho} \), if it is not zero. Said in another way, the action of a \( X \) in the root space relative to \( \alpha \) can be interpreted as a translation in the weight diagram shifting each of the dots (corresponding to the weights in the weight diagram) over by \( \alpha \). Hence the geometric meaning of the Theorem is that one can reach \( -\lambda \) from \( \lambda \) in at most 4 steps, in the real case (and in 2 steps in the other cases).

We shall now give the proof of (i). The other part is similar and actually easier. Recall that if \( \phi_\lambda \) is real, \( k(\lambda) \) is even. Thus if at least 6 steps would be necessary to go from \( -\lambda \) to \( \lambda \), by symmetry reasons there would exist a
weight $\mu$ lying on the wall of $s_0$ which is 3 steps far from both $-\lambda$ and $\lambda$. So any $v_\mu \in V_\mu$ could not belong to $U^2(t^C) \cdot v_\lambda + U^2(t^C) \cdot v_{-\lambda}$. But by Lemma 2.2, $U^2(t^C) \cdot v_\lambda + U^2(t^C) \cdot v_{-\lambda} = V_\lambda$.

Remark. Theorem 2.1 implies that if $\phi_\lambda : K \to O(V)$ is any representation of real type belonging to $(\mathfrak{e}_2)$, then $K$ has at most 4 simple factors and that if $\phi_\lambda$ is symplectic or complex, then $K$ has at most 2 simple factors. These results were proved in [5].

Observe also that Theorem 2.1 generalizes to (irreducible, faithful) orthogonal representations of class $(\mathfrak{c}_d)$. In that case we have that, if $\phi_\lambda$ is of real type, $k(\lambda) \leq 2d$ and, if $\phi_\lambda$ is of complex or symplectic type, $k(\lambda) \leq d$.

3. Symmetric weights

Wang and Ziller found very strong connections between the isotropy representations of irreducible symmetric spaces and their highest weights [16], cf. also [14]. In this Section we want to relate them with the conditions of symmetry of the highest weight and with the maximality of $k(\lambda)$.

As a start we give the following

DEFINITION [6, p. 128]. A highest weight $\lambda$ of a simple Lie algebra $\mathfrak{g}$ is called symmetric if all non-zero $\langle \lambda, \alpha \rangle$, $\alpha \in \Sigma^+$ are equal. Here $\langle , \rangle$ is a (uniquely defined up to a simple factor) $\mathfrak{g}$-invariant inner product on $\mathfrak{g}$. A highest weight of a semisimple $\mathfrak{g}$ is called symmetric if it is symmetric for each of its constant factors.

Note that, for semisimple $\mathfrak{g}$ one can rescale on each simple factor a $\mathfrak{g}$ invariant inner product $\langle , \rangle$ on $\mathfrak{g}$ so that all non-zero $\langle \lambda, \alpha \rangle$, $\alpha \in \Sigma^+$ are equal.

Next, we recall the results in [16] we refer to.

The first establishes a condition for the highest weight of the (complex) isotropy representation of a compact irreducible symmetric space.

THEOREM 3.1 [16]. Let $M = G/K$ be a compact irreducible symmetric space with $G$ the identity component of the full isometry group of $M$ and without euclidean factors. Let $B$ be the negative of the Killing form of $G$ and $\lambda$ the highest weight of its (complex) isotropy representation. Then

$$B(\lambda, \lambda) = 2B(\lambda, \alpha),$$

for any positive root $\alpha$ of $\mathfrak{g}$ such that $B(\lambda, \alpha) \neq 0$ and $2\lambda - \alpha$ is not a root.

As a consequence we get that, if $2\lambda - \alpha$ is not a root for any root $\alpha$ such that $B(\lambda, \alpha) \neq 0$, then the highest weight of an s-representation $\lambda$ is symmetric.

Remark. If $\phi$ is of real type and $k(\lambda)$ is 4, then $2\lambda - \alpha$ is never a root, [6, proof of Lemma 11, p. 1311].
If $\phi$ is of complex type and $k(\lambda)$ is 2, then $\lambda + \lambda^* = \lambda - \rho(\lambda)$ is never a root. This follows immediately since $\lambda + \lambda^* = \beta_1 + \beta_j$ for some $\beta_i, \beta_j \in \mathfrak{c}$ and the sum of two strongly orthogonal roots is never a root.

Wang and Ziller obtained a refinement of the above formula (3) for Hermitian and quaternionic symmetric spaces. Namely:

(i) If $G/H \cdot U(1)$ is Hermitian symmetric other than the complex projective spaces, with isotropy representation $[\phi_{\lambda}]_H = [\phi_{\lambda'} \otimes e^{i\theta}]_H$, then $B(\lambda, \lambda^*) = 0$ and $B(\lambda', \lambda') + B(\lambda', \lambda^*) = 2B(\lambda', \alpha)$, where $\alpha$ is any simple root of $H$ with $B(\lambda', \alpha) \neq 0$.

(ii) If $G/H \cdot Sp(1)$ is quaternionic symmetric other than the quaternionic projective spaces, with isotropy representation $\phi \otimes C = \phi_{\lambda} = \phi_{\lambda'} \otimes v_2$ or $\phi \otimes C = \phi_\lambda \otimes \phi_{\lambda'}^* = [\phi_{\lambda'} \otimes \phi_{\lambda'}^*] \otimes v_2$, then $B(\lambda', \lambda') = (3/2)B(\lambda', \alpha)$, where $\alpha$ is any simple root of $H$ with $B(\lambda', \alpha) \neq 0$.

Here and below $v_2$ is the two dimensional representation of $Sp(1)$.

Conversely Wang and Ziller proved that the above identities actually characterize the isotropy representations of irreducible symmetric spaces, among all irreducible representations of compact Lie groups. Namely

**Theorem 3.2** [16]. Let $\phi_\lambda : K \to O(V)$ be a faithful irreducible complex representation of a semisimple, compact, connected Lie group $K$ with highest weight $\lambda$, and let $\langle , \rangle$ be a $\mathfrak{k}$-invariant inner product on $\mathfrak{k}$.

(a) If $\phi_\lambda$ is of real type and $\langle \lambda, \lambda \rangle = 2\langle \lambda, \alpha \rangle$, for every simple root $\alpha$ of $K$ with $\langle \lambda, \alpha \rangle \neq 0$ then $\phi_\lambda^R$ is the isotropy representation of a compact, simply connected, irreducible symmetric space, except in the case $(G_2, \phi_7)$, where $\phi_7$ is the 7-dimensional representation of $G_2$.

(b) If $\phi_\lambda$ is of complex type (hence $\phi_\lambda \neq \phi_\lambda^*$) and $\langle \lambda, \lambda \rangle + \langle \lambda, \lambda^* \rangle = 2\langle \lambda, \alpha \rangle$, for every simple root $\alpha$ of $K$ with $\langle \lambda, \alpha \rangle \neq 0$ then $(K \cdot U(1), [\phi_\lambda \otimes e^{i\theta}]_H)$ is the isotropy representation of a compact, simply connected, irreducible hermitian symmetric space.

(c) If $\phi_\lambda$ is of symplectic type and $\langle \lambda, \lambda \rangle = 3/2\langle \lambda, \alpha \rangle$, for every simple root $\alpha$ of $K$ with $\langle \lambda, \alpha \rangle \neq 0$ then $(K \cdot Sp(1), \phi_\lambda \otimes v_2)$ is the isotropy representation of a compact, simply connected, irreducible quaternionic symmetric space.

We suppose now that $\lambda$ is symmetric and $k(\lambda)$ assumes the maximal value for representations of class $(0_2)$.

The following Lemma, together with Theorem 3.2 above, yields a first proof of Theorem 1.1.

**Lemma 3.1.** Let $\phi_\lambda$ be in the same hypothesis as in Theorem 3.2.

(a) If $\phi_\lambda$ is of real type, $k(\lambda) = 4$ and $\lambda$ is symmetric then $\langle \lambda, \lambda \rangle = 2\langle \lambda, \alpha \rangle$, for every positive root $\alpha$ of $K$ with $\langle \lambda, \alpha \rangle \neq 0$.

(b) If $\phi_\lambda$ is of complex type, $k(\lambda) = 2$ and $\lambda$ is symmetric then $\langle \lambda, \lambda \rangle + \langle \lambda, \lambda^* \rangle = 2\langle \lambda, \alpha \rangle$, for every positive root $\alpha$ of $K$ with $\langle \lambda, \alpha \rangle \neq 0$.

(c) If $\phi_\lambda$ is of symplectic type, $k(\lambda) = 3$ (thus in this case $\phi_\lambda \notin (0_2)!$) and $\lambda$ is symmetric then $\langle \lambda, \lambda \rangle = 3/2\langle \lambda, \alpha \rangle$, for every positive root $\alpha$ of $K$ with $\langle \lambda, \alpha \rangle \neq 0$. 

Note that in case (c) $\phi_2 \otimes v_2$ is of real type, its highest weight $\lambda'$ has $k(\lambda') = 4$. Thus (c) is a special case of (a).

Remark. We have the following generalization of Lemma 3.1. If $\phi_\lambda$ is of real type, $k(\lambda) = 2d$ and $\lambda$ is symmetric then $\langle \lambda, \lambda \rangle = d \langle \lambda, \alpha \rangle$, for every positive root $\alpha$ of $K$ with $\langle \lambda, \alpha \rangle \neq 0$. If $\phi_\lambda$ is of complex type, $k(\lambda) = d$ and $\lambda$ is symmetric then $\langle \lambda, \lambda \rangle + \langle \lambda, \lambda^* \rangle = d \langle \lambda, \alpha \rangle$, for every positive root $\alpha$ of $K$ with $\langle \lambda, \alpha \rangle \neq 0$.

Proof. (a) If $k(\lambda) = 4$, then $2\lambda = \beta_1 + \beta_2 + \beta_3 + \beta_4$, for $\beta_i \in \Theta$ not necessarily distinct. If $\lambda$ is symmetric, then $\langle \lambda, \alpha \rangle = \langle \lambda, \beta_i \rangle$ for any positive root $\alpha$ such that $\langle \lambda, \alpha \rangle \neq 0$ and any $i = 1, \ldots, 4$ fixed. Then

$$\langle \lambda, \lambda \rangle = \frac{1}{2}4\langle \lambda, \beta_i \rangle = 2\langle \lambda, \beta_i \rangle = 2\langle \lambda, \alpha \rangle.$$ 

(b) We remark first that $\lambda^* = -s_0(\lambda)$ [1, Chapitre VIII, Proposition 11, p. 13]. Thus, if $k(\lambda) = 2$, then $\lambda + \lambda^* = \lambda - s_0(\lambda) = \beta_1 + \beta_2$ for $\beta_i \in \Theta$ not necessarily distinct. Thus, since $\lambda$ is symmetric,

$$\langle \lambda, \lambda \rangle + \langle \lambda^*, \lambda \rangle = \langle \lambda + \lambda^*, \lambda \rangle = \langle \lambda, \beta_1 + \beta_2 \rangle = 2\langle \lambda, \beta_1 \rangle = 2\langle \lambda, \alpha \rangle,$$

for any positive $\alpha$ with $\langle \lambda, \alpha \rangle \neq 0$.

(c) If $k(\lambda) = 3$, $2\lambda = \beta_1 + \beta_2 + \beta_3$, for $\beta_i \in \Theta$ not necessarily distinct. If $\lambda$ is symmetric, then

$$\langle \lambda, \lambda \rangle = \frac{1}{2}3\langle \lambda, \beta_i \rangle = \frac{3}{2}\langle \lambda, \beta_i \rangle = \frac{3}{2}\langle \lambda, \alpha \rangle,$$

for any positive $\alpha$ with $\langle \lambda, \alpha \rangle \neq 0$. q.e.d.

Observe that in case (a) the highest weight $\mu$ of the representation $\phi_\gamma$ of $G_2$ has $k(\lambda) = 2$. Hence the conditions of Theorem 1.1 do not hold.

Thus Lemma 3.1 gives a proof of Theorem 1.1, using the results in [16]. However the proof in [16], when $\phi_\lambda$ is orthogonal and $K$ is not simple, is case by case and uses the classification of symmetric spaces. Thus we will give a conceptual proof for this case. If $\phi_\lambda$ is complex, we will show that $K$ is simple. Thus we will use Theorem 4.6 in [16] which has a conceptual proof (cf. also [9]).

As a start, we prove the following

**Lemma 3.2.** Let $\phi_\lambda$ be an irreducible representation, not of symplectic type, of a semisimple compact Lie algebra $\mathfrak{k}$ with $\lambda$ symmetric. Assume that $k(\lambda) = 4$, if $\phi_\lambda$ is of real type and $k(\lambda) = 2$, if $\phi_\lambda$ is of complex type. Then

(a) for any simple factor $\mathfrak{k}_i$ of $\mathfrak{k}$ there exists a unique simple root $\alpha_i^\prime$ which is not orthogonal to $\lambda$;

(b) any positive root of $\mathfrak{k}_i$ has either 0 or 1 as coefficient of $\alpha_i^\prime$ in its expression as a linear combination of simple roots;

(c) if $\lambda_i$ is the highest weight of $\phi_{\mathfrak{k}_i}$, then $\lambda_i$ is an integral multiple of a minuscule weight of $\mathfrak{k}_i$. Moreover the coefficient of proportionality equals $\lambda(H_{\alpha_i^\prime})$. 

Remark. More in general the present Lemma holds if \( k(\lambda) = 2d \) and \( \phi_\lambda \) is of real type or \( k(\lambda) = d \) and \( \phi_\lambda \) is of complex type.

Proof. (a) Let \( \alpha \) and \( \beta \) be simple roots (chosen in the same simple factor) such that \( \langle \lambda, \alpha \rangle \neq 0 \) and \( \langle \lambda, \beta \rangle \neq 0 \). Then, by [7, p. 266] there exists a minimal chain of simple roots \( \alpha_1 = \alpha, \alpha_2, \ldots, \alpha_r = \beta \) connecting \( \alpha \) and \( \beta \). Let \( \gamma = \alpha_1 + \cdots + \alpha_r \). Then \( \gamma \) is a root of \( \mathfrak{g} \) and \( \langle \lambda, \gamma \rangle \neq 0 \). In the real case we have that, since \( \lambda \) is symmetric

\[
\langle \lambda, \lambda \rangle = 2 \langle \lambda, \alpha \rangle = 2 \langle \lambda, \gamma \rangle.
\]

On the other hand, \( \langle \lambda, \lambda \rangle = 2 \langle \lambda, \alpha_1 \rangle + \cdots + 2 \langle \lambda, \alpha_r \rangle \geq 2 \langle \lambda, \alpha \rangle + 2 \langle \lambda, \beta \rangle = 2 \langle \lambda, \lambda \rangle \), which is clearly a contradiction.

In the complex case one gets similarly a contradiction from \( \langle \lambda + \lambda^*, \lambda \rangle = 2 \langle \lambda + \lambda^*, \lambda \rangle \).

(b) We know by (a) that for any simple factor \( \mathfrak{t}_i \) of \( \mathfrak{g} \) there exists a unique simple root \( \alpha_i^\prime \) not orthogonal to \( \lambda \). On the other hand, if we denote by \( \tilde{\alpha}^i = \sum a_k \alpha_k^\prime \) the maximal root of \( \mathfrak{t}_i \), we have

\[
\langle \lambda, \alpha_i^\prime \rangle = 2 \langle \lambda, \tilde{\alpha}^i \rangle = \langle \lambda, \tilde{\alpha}^i \rangle.
\]

Then \( \tilde{\alpha}^i \) has coefficient 1 in \( \alpha_i^\prime \) in its expression as a linear combination of simple roots. But, in general, for any root of \( \mathfrak{t}_i \), \( \beta^i = \sum b_k \alpha_k^\prime \) we have \( b_k \leq a_k \), thus we get (b).

(c) By (a) \( \lambda_i = m_i \omega_j \), where \( m_i \) is an integer and \( \omega_j \) is the fundamental weight of \( \mathfrak{t}_i \), corresponding to the simple root \( \alpha_j^\prime \). On the other hand, by (b) the maximal root \( \tilde{\alpha}^i \) has coefficient 1 in \( \alpha_j^\prime \). Thus by [1, Chapitre VIII, Proposition 8, p. 128] \( \omega_j \) is a minuscule weight. Moreover \( \langle H_{s_j^\prime}, \lambda_i \rangle = \langle H_{s_j^\prime}, \lambda \rangle = m_i \). q.e.d.

Notation. In any simple factor \( \mathfrak{t}_i \) the system of simple roots will be denoted by \( \alpha_{i_1}^\prime, \ldots, \alpha_{i_l}^\prime \), with \( l \) the rank of \( \mathfrak{t}_i \). The unique simple root not orthogonal to \( \lambda \) will be denoted by \( \alpha_{i_j}^\prime \).

Observe that \( k(\lambda_j) = m_j k(\omega_j) \).

Next we deal with the case of representations of real type.

If \( k(\lambda) = 4 \), \( \lambda(H_{s_j^\prime}) \leq 4 \) and \( \lambda_i \) is a sum of \( k(\lambda_i) \) strongly orthogonal positive roots, so we have the following cases:

1. \( m_i = 1 \) for any \( i = 1, \ldots, l \). Then \( \lambda \) is a sum of minuscule weights (each for any simple factor).
2. \( m_i = 2 \) for one \( i \). Then if \( k(\omega_j) = 2 \), \( \mathfrak{g} \) is simple of types \( B_l \) or \( D_l \) (and \( \lambda = 2\omega_1 \); here and below we use the notation of [1] for the fundamental weights). If \( k(\omega_j) = 1 \), then, if \( \mathfrak{g} \) has 2 simple factors, \( \mathfrak{g} \) is of type \( A_1 + B_l \) or \( A_1 + D_l \) and \( \lambda = 2\omega_1 + \omega_1 \), if \( \mathfrak{g} \) has 3 simple factors it is of type \( A_1 + A_1 + A_1 \) and \( \lambda = 2\omega_1 + \omega_1 + \omega_1 + \omega_1 \).
3. \( m_i = 3 \) for one \( i \). Then \( k(\omega_j) = 1 \) and \( \mathfrak{g} \) is of type \( A_1 + A_1 \) with \( \lambda = 3\omega_1 + \omega_1 \).
4. \( m_i = 4 \) for a (unique) \( i \). Then \( \mathfrak{g} \) is simple and of type \( A_1, \lambda = 4\omega_1 \).
Like in [16], we give the following decomposition of the second exterior power of $\phi_{\lambda}$:

$$\Lambda^2 \phi_{\lambda} = \text{ad}_t \oplus \chi,$$

where $\chi$ is the isotropy representation of $SO(V_{\lambda})/\phi_{\lambda}(K)$. We will compute the irreducible components of $\chi$ (with respect to $K$).

Our goal is to prove that there exists a non trivial $\mathfrak{l}$-invariant curvature tensor $R$ on $V_{\lambda}$ with values in $\phi_{\lambda}(\mathfrak{l}) \subset \mathfrak{so}(V_{\lambda}) \cong \Lambda^2 V_{\lambda}$. Then, as we will explain below, the classical Cartan's construction yields a conceptual proof of Theorem 1.1.

By a $\mathfrak{l}$-invariant curvature tensor $R$ on $V_{\lambda}$ we mean a $(4,0)$ tensor on $V_{\lambda}$ which is invariant by the naturally induced action of $\mathfrak{l}$ on the space of $(4,0)$ tensors and has the same algebraic properties as the Riemannian curvature tensor. It can be easily seen that such an object identifies with a $\mathfrak{l}$-invariant element of the second symmetric power of $\Lambda^2 V_{\lambda}$, $S^2 \Lambda^2 V_{\lambda}$. More precisely, if $\mathcal{R}$ denotes the space of such curvature tensors we have the decomposition

$$S^2 \Lambda^2 V_{\lambda} = \Lambda^4 V_{\lambda} \oplus \mathcal{R},$$

with $\mathcal{R}$ the space of $\mathfrak{l}$-invariant curvature tensors.

The Cartan's construction can be summarized as follows (cf. [16, Lemma 4.1]). Let $R$ be a non trivial $\mathfrak{l}$-invariant curvature tensor with values in $\phi_{\lambda}(\mathfrak{l}) \subset \mathfrak{so}(V_{\lambda}) \cong \Lambda^2 V_{\lambda}$. Then the vector space $\mathfrak{g} = \mathfrak{l} \oplus V_{\lambda}$ can be made into a Lie algebra by defining

$$[X, v] = -[v, X] = \phi_{\lambda}(X)(v), \quad X \in \mathfrak{l}, v \in V_{\lambda},$$

$$[v, w] = Y \in \mathfrak{l}, \quad \text{where} \quad \phi_{\lambda}(Y) = -R(v, w), \quad u, v \in V_{\lambda}.$$ 

Our strategy is then to show that our strong assumptions on $\lambda$ (i.e. $\lambda$ symmetric and $k(\lambda) = 4$) imply that there exists such a curvature tensor. To do this we deduce from the decomposition of $\Lambda^2 V_{\lambda}$ into irreducible summands, the dimensions of the space of $\mathfrak{l}$-invariant elements in $S^2 \Lambda^2 V_{\lambda} = \Lambda^4 V_{\lambda} \oplus \mathcal{R}$. Roughly, we show that dim $\mathcal{R}$ is big enough so that a linear combination of $\mathfrak{l}$-invariant curvature tensors yields a non trivial one having values in $\phi_{\lambda}(\mathfrak{l})$.

This technique was first used by Kostant if the space of $\mathfrak{l}$-invariant tensors in $\Lambda^4 V_{\lambda}$, $(\Lambda^4 V_{\lambda})^\mathfrak{l}$, vanishes [11], and in case $\chi$ is irreducible in [16].

The following lemma gives the decomposition of $\Lambda^2 \phi_{\lambda}$ into irreducible components, by describing the highest weights of the irreducible components of $\chi$.

**Lemma 3.3.** The highest weights of the irreducible components of $\chi$ are $2\lambda - \alpha_j^\vee$, where $\alpha_j^\vee$ is the unique simple root in the simple factor $\mathfrak{l}$, which is not orthogonal to $\lambda$.

Note that, as a consequence, we have that, if $\mathfrak{l}$ has $l \leq 4$ simple factors then $\text{ad}_t$ has $l$ irreducible components and $\chi$ has $l$ irreducible components.
Proof. If $\lambda$ is not a sum of minuscule weights (each for any simple factor), one can check the Lemma directly. This corresponds to the few cases mentioned above ((2), (3) and (4)).

Next we consider the case of $\lambda$ sum of minuscule weights. Recall that this implies that, for any simple factor $f$, the weights of $\phi |_{f}$ are a single orbit of the Weyl group of $f$. In particular, all weights have multiplicity one and have the same length.

If $\langle \lambda, \alpha \rangle \neq 0$, then $\lambda - \alpha$ is a weight and $\langle \lambda, \lambda \rangle = \langle \lambda - \alpha, \lambda - \alpha \rangle$. Thus $\langle \lambda, \lambda \rangle = \langle \alpha, \alpha \rangle$. Moreover $\lambda - 2\alpha$ is never a weight. If, in addition, $\lambda - \alpha - \beta$ is a weight with $\alpha \neq \beta$ in the same simple factor, $\alpha + \beta$ not a root and $\langle \lambda, \alpha \rangle = \langle \lambda, \beta \rangle \neq 0$, then an easy computation shows that $\langle \alpha, \beta \rangle = 0$, i.e., that $\alpha$ and $\beta$ are strongly orthogonal. So we may assume that $\alpha, \beta \in \mathcal{O}$.

Hence, if $2\lambda = \beta_1 + \beta_2 + \beta_3 + \beta_4$, the possible highest weights of $\chi$ can be of the types

$$2\lambda - \alpha_1', \quad 2\lambda - \beta_r, \quad 2\lambda - \beta_r - \beta_s.'$$

On the other hand, $2\lambda - \alpha_1'$ has clearly multiplicity one in $\chi$. We are thus left to show that for $\beta_r, \beta_s$ both in $f$, $2\lambda - \beta_r$ and $2\lambda - \beta_r - \beta_s$ belong to the irreducible component $V_{2\lambda - \alpha_1'}$ of $\chi$ having $2\lambda - \alpha_1'$ as highest weight. But this is clear, since they have the same multiplicity in $\Lambda^2 \phi_\lambda$ and in $V_{2\lambda - \alpha_1'}$. q.e.d.

Next we observe that since $ad_{\mathfrak{f}} \oplus \chi$ has $2l$ irreducible components, by Schur's lemma, the space of $f$-invariant tensors on $S^2 \Lambda^2 V_\lambda = S^2 (ad_{\mathfrak{f}} \oplus \chi)$ has dimension $2l$, i.e.

$$\dim(S^2(ad_{\mathfrak{f}} \oplus \chi))^f = 2l.$$ 

Thus a $f$-invariant curvature tensor $R \in \mathfrak{g}$ can be written as a matrix of the following form with respect to the decomposition of $\Lambda^2 V_\lambda = ad_{\mathfrak{f}} \oplus \chi$ into irreducible summands

$$R = \begin{pmatrix} a_1 \mathbf{1} & \cdots & a_l \mathbf{1} & 0 \\ \cdots & b_1 \mathbf{1} & \cdots & b_l \mathbf{1} \\ 0 & \cdots & b_1 \mathbf{1} \end{pmatrix},$$

where the $a_l$ are in the $ad_{\mathfrak{f}}$ component and the $b_j$ in the $\chi$ component. (Actually, we identify an element of $S^2 \Lambda^2 V_\lambda$ with the corresponding symmetric endomorphism of $\Lambda^2 V_\lambda$.)

Hence, in order to have a non trivial invariant curvature tensor with values in $f$ we need to have at least $l + 1$ independent invariant curvature tensors in $\mathfrak{g}$, i.e., we have to prove

$$\dim \mathfrak{g}^f \geq l + 1 \quad \text{or} \quad \dim(\Lambda^4 V_\lambda)^f \leq l - 1.$$
To this purpose we proceed like in [16, pp. 308–309]. We choose in each simple factor a Chevalley basis, take weight vectors \( \upsilon^\chi \) and \( \upsilon^{-\chi} \) such that \( \langle \upsilon^\chi, \upsilon^{-\chi} \rangle = 1 \) and set

\[
v_{\chi^{-\chi}} = \phi_\chi(X_{\chi^{-\chi}}) \upsilon_\chi, \quad v_{\chi^{\chi}} = \phi_\chi(X_{\chi^{\chi}}) \upsilon_\chi.
\]

Then (by [16, p. 308]) we have \( \langle v_{\chi^{-\chi}}, v_{\chi^{\chi}} \rangle = -\langle \lambda, \chi \rangle \). We denote by \( pr_t(pr_t) \) the orthogonal projection from \( \Lambda^2 V_\chi \) to \( \mathfrak{f}(\mathfrak{t}) \) via the embedding \( \phi_\chi \). As in [16] one can prove that

\[
pr_t(v_{\chi^{-\chi}} \wedge v_{-\chi}) = H_\lambda^1,
\]

\[
pr_t(v_{\chi^{-\chi}} \wedge v_{\chi^{\chi}}) = -\langle \lambda, \chi \rangle X_{\chi^{-\chi}},
\]

\[
pr_t(v_{\chi^{-\chi}} \wedge v_{\chi^{\chi}}) = -\langle \lambda, \chi \rangle X_{\chi^{\chi}},
\]

\[
pr_t(v_{\chi^{-\chi}} \wedge v_{-\chi}) = \langle \lambda, \chi \rangle X_{\chi^{-\chi}},
\]

\[
pr_t(v_{\chi^{-\chi}} \wedge v_{\chi^{\chi}}) = 0.
\]

Let \( \eta \in (\Lambda^4 V_\chi)^t \), then it can be written as a matrix in the same form as \( (R) \).

Since \( \eta \) is a 4-form we have that the quantities

\[
\eta(v_\lambda, v_{\chi^{-\chi}}, v_{-\chi}, v_{\chi^{\chi}}) = -b_i \langle \lambda, \chi \rangle,
\]

\[
\eta(v_\lambda, v_{\chi^{-\chi}}, v_{-\chi}, v_{\chi^{\chi}}) = -a_i \langle \lambda, \chi \rangle^2 + b_i \langle \lambda, \chi \rangle^2 (1 + \langle \lambda, \chi \rangle),
\]

\[
\eta(v_\lambda, v_{-\chi}, v_{\chi^{\chi}}, v_{\chi^{-\chi}}) = \sum a_i \langle H_\lambda^i, H_{\chi^{\chi}} \rangle \langle \lambda, \chi \rangle
\]

are all equal. The equality between the first and the second gives rise to the equations

\[
b_i = a_i \langle \lambda, \chi \rangle - b_i (1 + \langle \lambda, \chi \rangle), \quad i = 1, \ldots, l
\]

i.e., to \( l \) independent equations.

The equality between the second and the third yields at least one more linearly independent condition. This proves that \( \dim(\Lambda^4 V_\chi)^t \leq l - 1 \).

Now we deal with the case of representations of complex type.

Let \( \phi_\chi \) be an irreducible representation of complex type of a compact semisimple Lie algebra \( \mathfrak{f} \). Consider the splitting

\[
\phi_\chi \otimes \phi_\chi^* = \mathbf{1} \oplus \text{ad} \oplus \chi,
\]

where \( \mathbf{1} \) is the trivial representation and \( \chi \) is the isotropy representation of the homogeneous space \( SU(V_\chi)/\phi_\chi(K) \).

**Lemma 3.4.** Let \( \phi_\chi \) be an irreducible representation of complex type of a compact semisimple Lie algebra \( \mathfrak{f} \) with \( \chi \) symmetric and \( k(\chi) = 2 \). Then

(a) either \( \mathfrak{f} \) is simple or \( \mathfrak{f} = \mathfrak{su}(m) \oplus \mathfrak{su}(m') \) and \( \phi_\chi \) is the external tensor product of the standard representation of \( \mathfrak{su}(m) \) on \( \mathbb{C}^m \) and the dual of the standard representation of \( \mathfrak{su}(m') \) on \( \mathbb{C}^{m'} \);
(b) if \( \mathfrak{g} \) is simple, then \( \chi = \phi_{\lambda + \lambda^*} \) (in particular, \( \chi \) is irreducible), \( \Lambda^2\phi_\lambda = \phi_{2\lambda - \alpha} \) (with \( \alpha_i \) is the unique simple root not orthogonal to \( \lambda \)).

**Proof.** (a) We already know that either \( \mathfrak{g} \) is simple or it has two simple factors. In the latter case, \( \phi_\lambda \) is the external tensor product of two representations \( \phi_{\lambda_i} \) of the two simple factors \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \), with \( k(\lambda_i) = 1 \), \( \lambda_i \) symmetric and \( \lambda + \lambda^* = \beta_1 + \beta_2 \). Thus, by the classification of simple Lie algebras, we must have \( \mathfrak{g} \) of type \( A_l \).

(b) By Lemma 3.2 (c), if \( \mathfrak{g} \) is simple, then \( \lambda = m_i \omega_i \) is an integral multiple of the minuscule weight \( \omega_i \) corresponding to the unique simple root \( \alpha_i \) not orthogonal to \( \lambda \). Moreover \( m_i = \lambda(H_{\alpha_i}) = 1 \). Indeed, if \( \lambda - 2\alpha \) would be a weight we would have that \( \lambda + \lambda^* = 2\alpha \) and thus the only possibility would be that \( \mathfrak{g} \) is of type \( A_l \) and \( \lambda = 2\omega_1 \), but in this case \( \phi_\lambda \) is not of complex type. Thus \( \lambda = \omega \) and all weights have multiplicity one and the same length.

If \( \langle \lambda, \alpha \rangle \neq 0 \), then \( \lambda - \alpha \) is a weight and \( \langle \lambda, \lambda \rangle = \langle \lambda - \alpha, \lambda - \alpha \rangle \). Thus \( \langle \lambda + \lambda^*, \lambda \rangle = \langle \alpha, \alpha \rangle \). Moreover \( \lambda - 2\alpha \) is never a weight. If, in addition, \( \lambda - \alpha - \beta \) is a weight with \( \alpha + \beta \) not a root and \( \langle \lambda, \alpha \rangle = \langle \lambda, \beta \rangle \neq 0 \), then an easy computation shows that \( \langle \alpha, \beta \rangle = 0 \), i.e., that \( \alpha \) and \( \beta \) are strongly orthogonal. So we may assume that \( \alpha, \beta \in \mathcal{O} \).

Then if \( \lambda + \lambda^* = \beta_1 + \beta_2 \), the possible highest weights of \( \phi_\lambda \otimes \phi_\lambda^* \) are of the form

\[
\lambda + \lambda^* \quad \text{or} \quad \lambda + \lambda^* - \beta_r \quad \text{or} \quad \lambda + \lambda^* - \beta_r - \beta_s.
\]

On the other hand \( \lambda + \lambda^* \) is a highest weight of \( \chi \) with multiplicity one in \( \phi_\lambda \otimes \phi_\lambda^* \) and we have that \( \lambda + \lambda^* - \beta_1 = \beta_2 \) and \( \lambda + \lambda^* - \beta_1 - \beta_2 = 0 \) are highest weights of \( \phi_\lambda \otimes \phi_\lambda^* \) which belong to \( ad_\mathfrak{g} \) and \( I \) respectively. Then, proceeding like in the proof of Lemma 3.3 it is possible to prove that \( \chi \) is irreducible with highest weight \( \lambda + \lambda^* \).

The highest weight \( 2\lambda - \alpha \) has multiplicity one in \( \Lambda^2\phi_\lambda \). Hence, like in the proof of Lemma 3.3, one gets that \( \Lambda^2\phi_\lambda = \phi_{2\lambda - \alpha} \).

We can now apply Theorem 4.6 in [16]. This completes our proof of Theorem 1.1.

**Final remark.** As already observed many properties of irreducible representations with \( \lambda \) symmetric and \( k(\lambda) = 2d \), in the real case and \( k(\lambda) = d \) otherwise, are similar to the ones in the special case \( d = 2 \) (Lemma 3.1 and 3.2). It would be interesting to give a geometric characterization of these representations in the general case.

**References**


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