Cohomology and Stiefel-Whitney Classes of Flat Manifolds

joint work with Roberto Miatello and Juan Pablo Rossetti, Cordoba, Argentina

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1. Compact flat manifolds and Bieberbach groups
   - Flat manifolds and spectra
   - Bieberbach groups
   - Cohomology of Bieberbach groups

2. The Lyndon-Hochschild-Serre Spectral Sequence
   - The Charlap-Vasquez method
   - Use of the LHS Spectral Sequence
   - Second Stiefel-Whitney class

3. Topology and spectra
   - Cohomology and spectral properties
   - Stiefel-Whitney classes and spectral properties
Flat manifolds and spectra

\( M, M' \) are \( p \)-isospectral \( \iff \) have the same spectrum with respect to the Hodge Laplacian \( \Delta_p \) acting on \( p \)-forms.

For flat manifolds \( \mathbb{R}^n/\Gamma \) of diagonal type equal Sunada numbers (combinatorial property on \( \Gamma \)) \( \implies \) \( p \)-isospectrality for all \( p \) \((0 \leq p \leq n)\)

\( M, M' \) \( p \)-isospectral \( \implies \) \( b_p(M) = b_p(M') \)

(the \((\mathbb{Z})\)-Betti number \( b_p \) equals the multiplicity of the 0 eigenvalue of \( \Delta_p \))

So, the torsionfree part cannot be distinguished \( \implies \) it is not so easy to exhibit \( p \)-isospectral manifolds for all \( p \) having different cohomological properties.

We construct for instance \( M, M', p \)-isospectral for all \( p \) with

- \( H^1(M, \mathbb{Z}_2) \cong H^1(M', \mathbb{Z}_2) \) but \( H^2(M, \mathbb{Z}_2) \not\cong H^2(M', \mathbb{Z}_2) \)
- same \((\mathbb{Z}_2)\)-cohomology but such that \( w_2(M) \neq 0 \) and \( w_2(M') = 0 \)
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Examples in column notation

**Hantzsche-Wendt 3-manifold or didicosm**

$M_\Gamma$: flat manifold of dimension $n = 3$,

Holonomy group of $\Gamma = \mathbb{Z}_2^2$,

generated by $B_1 = \text{diag}(1, -1, -1)$, $B_2 = \text{diag}(-1, 1, -1)$,

$b_1 = \frac{e_1 + e_3}{2}$, $b_2 = \frac{e_1 + e_2}{2}$; i.e., $\Gamma = \langle B_1 L_{b_1}, B_2 L_{b_2}; L_{\mathbb{Z}^3} \rangle$.

$M_\Gamma$ is orientable since $\det B = 1$ for every $BL_b \in \Gamma$.

In column notation

<table>
<thead>
<tr>
<th>$B_1$</th>
<th>$B_2$</th>
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<tbody>
<tr>
<td>$\frac{1}{2}$</td>
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\hline
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\begin{array}{ccc}
B_1 & B_2 & B_1B_2 \\
\hline
1 & -1 & -1 \\
-1 & 1 & -1 \\
-1 & -1 & 1
\end{array}
\]
**zo₂-class polynomial**

\[ \Gamma: \text{Bieberbach group of diagonal type and holonomy group } \mathbb{Z}_2^k = \langle B_1, \ldots, B_k \rangle. \]

\[ \bar{\beta} \in H^2(\mathbb{Z}_2^k, \Lambda^* \otimes \mathbb{Z}_2) \cong (H^2(\mathbb{Z}_2^k, \mathbb{Z}_2))^n \]

The components \( \bar{\beta}_\ell \) of \( \bar{\beta} \) are homogeneous polynomials of degree two called the **\( \mathbb{Z}_2 \)-class polynomials** of \( \Gamma \).

**Proposition**

\[ \bar{\beta}_\ell = \sum_{i : B_i e_\ell = e_\ell} x_i^2 + \sum_{i : b_{i\ell} = \frac{1}{2}} \sum_{j \neq i} x_i x_j, \]

where \( e_1, \ldots, e_n \) is the standard basis of \( \mathbb{R}^n \).
The components $\bar{\beta}_\ell$ of $\bar{\beta}$ are homogeneous polynomials of degree two called the $\mathbb{Z}_2$-class polynomials of $\Gamma$.

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Lyndon-Hochschild-Serre spectral sequence

Γ is an extension of $F$ by $\Lambda$, i.e., $0 \to \Lambda \to \Gamma \to F \to 1$

$$E^p,q_2 \Rightarrow \text{H}^{p+q}(\Gamma, R)$$

with

$$E^p,q_2 \cong \text{H}^p(F, \text{H}^q(\Lambda, R)),$$

(the coefficient ring $R$ is regarded as a trivial $\Gamma$-module and $p, q \geq 0$)
LHS spectral sequence for Bieberbach groups

For $\Gamma$ Bieberbach group of diag. type with holonomy group $\mathbb{Z}_2^k$

$$E_2^{p,q} = H^p(\mathbb{Z}_2^k, \mathbb{Z}_2) \otimes \wedge^q(\mathbb{Z}_2^*)$$

and their dimensions are given by

\[
\begin{array}{c|cccc}
  q & 2 & 1 & 0 & \\
  \hline
  2 & \binom{n}{2} & k \binom{n}{2} & \left(\frac{k+1}{2}\right)n & \\
  1 & n & kn & \left(\frac{k+1}{2}\right)n & \\
  0 & 1 & k & \left(\frac{k+1}{2}\right) & \\
  \hline
  0 & 1 & 2 & 3 & p
\end{array}
\]
The differential $d_{2}^{p,q}$ in low dimensions

Let $\varepsilon^{1}, \ldots, \varepsilon^{n}$ be a basis of $\Lambda^{*} \otimes \mathbb{Z}_{2} \cong (\mathbb{Z}_{2}^{n})^{*}$,

\[ d_{2}^{0,1} \colon E_{2}^{0,1} \cong H^{1}(\mathbb{Z}_{n}, \mathbb{Z}_{2}) \cong (\mathbb{Z}_{2}^{n})^{*} \rightarrow E_{2}^{2,0} \cong H^{2}(\mathbb{Z}_{2}, \mathbb{Z}_{2}) \]

\[ d_{2}^{0,1} \varepsilon^{i} = \overline{\beta}_{i}, \quad i = 1, \ldots, n \]

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\[ d_{2}^{1,1}(x_{i} \otimes \varepsilon_{j}) = x_{i} \cup \overline{\beta}_{j}, \quad i = 1, \ldots, k, \quad j = 1, \ldots, n \]

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\[ d_{2}^{0,2}(\varepsilon^{i} \wedge \varepsilon^{j})(\varepsilon^{k}) = \delta_{ik} \overline{\beta}_{j} + \delta_{jk} \overline{\beta}_{i}, \quad i, j, k = 1, \ldots, n \]

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<thead>
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<th></th>
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<tr>
<td>2</td>
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Examples of application of the Charlap-Vasquez method

Computation of $H^1(\Gamma, \mathbb{Z}_2)$:

Theorem

Let $M$ be an $n$-dimensional compact flat manifold with diagonal holonomy $\mathbb{Z}_2^k$. Then

$$\dim H^1(M, \mathbb{Z}_2) = n - \text{rank } d_{0,1}^{0,1} + k.$$ 

Note that $\text{rank } d_{0,1}^{0,1}$ coincides with the number of linearly independent $\mathbb{Z}_2$-class polynomials $\bar{\beta}_\ell$, $\ell = 1, \ldots, n.$
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![Diagram showing the computation of $H^1(\Gamma, \mathbb{Z}_2)$]

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Second Stiefel-Whitney class

\[ \Gamma \xrightarrow{r} F \cong \mathbb{Z}_2^k \xrightarrow{i} D(n) \cong \mathbb{Z}_2^n \hookrightarrow O(n) \]

\( (D(n) \cong \mathbb{Z}_2^n): \text{diagonal matrices in } O(n) \)

induces a map of \( M \) into \( BO(n) = \text{classifying map for } TM \).

Let \( x_1, \ldots, x_k \) be a basis of \( H^1(\mathbb{Z}_2^k, \mathbb{Z}_2) \) and let \( x'_1, \ldots, x'_n \) be the standard basis of \( H^1(D(n), \mathbb{Z}_2) \). The classes

\[ \omega_\ell = i^*(x'_\ell) = \sum_{m} a_{m\ell} x_m, \]

are called the 2-weights of the map \( i \).

\[ \sigma_j(\omega_1, \ldots, \omega_n) = j\text{-th elem symmetric function in } \omega_1, \ldots, \omega_n. \]

Proposition

Let \( \Gamma \xrightarrow{r} F = \mathbb{Z}_2^k \) be the projection map. Then

\[ w_j(M) = r^* \sigma_j(\omega_1, \ldots, \omega_n). \]
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By the LHS Exact Sequence

$$\cdots \rightarrow H^1(\Lambda, \mathbb{Z}_2) \xrightarrow{d^0,1} H^2(F, \mathbb{Z}_2) \xrightarrow{r^*} H^2(\Gamma, \mathbb{Z}_2)$$

$$\implies \ker r^* = \text{Im } d^0,1 = \text{sums of } \mathbb{Z}_2\text{-class polynomials}$$

So

**Theorem**

Let $M_\Gamma$ be an $n$-dim compact flat manifold with diagonal holonomy $\mathbb{Z}_2^k$. Then,

$$w_2 \neq 0 \iff \sigma_2(\omega_1, \ldots, \omega_n) \text{ is not a sum of } \mathbb{Z}_2\text{-class polyn.}$$
Second Stiefel-Whitney class

By the LHS Exact Sequence

\[ \cdots \rightarrow H^1(\Lambda, \mathbb{Z}_2) \overset{d^0,1_2}{\rightarrow} H^2(F, \mathbb{Z}_2) \overset{r^*}{\rightarrow} H^2(\Gamma, \mathbb{Z}_2) \]

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\( M_{\Gamma} \) \( n \)-dim compact flat manifold with diagonal holonomy \( \mathbb{Z}_2^k \).

Then,

\( w_2 \neq 0 \iff \sigma_2(\omega_1, \ldots, \omega_n) \text{ is not a sum of } \mathbb{Z}_2\text{-class polyn.} \)
Second Stiefel-Whitney class

By the LHS Exact Sequence

\[ \cdots \rightarrow H^1(\Lambda, \mathbb{Z}_2) \xrightarrow{d_2^{0,1}} H^2(F, \mathbb{Z}_2) \xrightarrow{r^*} H^2(\Gamma, \mathbb{Z}_2) \]

\[ \Rightarrow \quad \ker r^* = \text{Im } d_2^{0,1} = \text{sums of } \mathbb{Z}_2\text{-class polynomials} \]

So

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Cohomology and spectral properties

- We consider all 4-dimensional flat manifolds of diagonal type with $F \equiv \mathbb{Z}_2^2$ or $F \equiv \mathbb{Z}_2^3$ and we show several isospectral or $p$-isospectral pairs, with $1 \leq p \leq 3$, having different $\mathbb{Z}_2$-cohomology groups and where some of them have different lengths of closed geodesics.

- We find, for $n = 5$, many isospectral pairs with $F \equiv \mathbb{Z}_2^4$ having different $H^2(M_\Gamma, \mathbb{Z}_2)$ and having the same $H^1(M_\Gamma, \mathbb{Z}_2)$ and the same holonomy representations. Such examples are not possible to obtain in dimension 4.

Example ($#g1, #g4$ in the CARAT (Aachen) list)

<table>
<thead>
<tr>
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Example ($#g_1, #g_4$ in the CARAT (Aachen) list)
We determine the $\mathbb{Z}_2$-cohomology of all GHW manifolds in dimensions 3, 4 and 5, listing all isospectral classes.

$GHW =$ generalized Hantzsche-Wendt manifolds: dimension $n$ flat manifolds having holonomy group $\mathbb{Z}_2^{n-1}$

$HW =$ orientable GHW

they are all rational homology spheres

**Table:** Cohomology classes of GHW manifolds in dimension 5.

<table>
<thead>
<tr>
<th>betti$_1$</th>
<th>betti$_2$</th>
<th>List of manifolds</th>
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<tbody>
<tr>
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<td>5</td>
<td>7, 14, 16, 21, 58, 61, 67, 69, 74, 77, 84, 85, 104, 105, 106, 107, 112, 115, 117, 118, 121, 122</td>
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<tr>
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<tr>
<td>5</td>
<td>10</td>
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some isospectral manifolds have the same colors
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some isospectral manifolds have the same colors
Stiefel-Whitney classes and spectral properties

- For $n = 4$, we exhibit $p$-isospectral pairs for all $p$, $M, M'$, that have the same $(\mathbb{Z}_2)$-cohomology but such that $w_2(M) \neq 0$ and $w_2(M') = 0$

See manifolds labelled $(1, 1, 0)$ and $(1, 0, 1)$ in the family $\mathcal{K}_4$:

<table>
<thead>
<tr>
<th></th>
<th>$B_1$</th>
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<td></td>
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<td>$1$</td>
<td>$1$</td>
</tr>
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**Table: Family \( \mathcal{K}_4 \)**

<table>
<thead>
<tr>
<th>((x,y,z))</th>
<th>(\text{betti}^{Z_2}_1 = \text{betti}^{Z_2}_3)</th>
<th>(\text{betti}^{Z_2}_2)</th>
<th>(w_2)</th>
<th>Sunada n.</th>
<th>isospectral pairs</th>
</tr>
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<tbody>
<tr>
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<td>6</td>
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<td>((1\ 0\ 0))</td>
<td>((2\ 1\ 0))</td>
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<tr>
<td>((1,0,0))</td>
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</tr>
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</table>

**Sunada numbers:** \(c_{s,t} = \text{number of elements in the holonomy } F \text{ of } M_\Gamma\), having exactly \(s\) 1’s in the diagonal (or column) and \(t\) \(\frac{1}{2}\)'s coming with those 1’s, \(0 \leq t \leq s \leq n\).